Percolation of even sites for enhanced random sequential adsorption

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Abstract

Consider random sequential adsorption on a chequerboard lattice with arrivals at rate 1 on light squares and at rate λ on dark squares. Ultimately, each square is either occupied, or blocked by an occupied neighbour. Colour the occupied dark squares and blocked light sites *black*, and the remaining squares *white*. Independently at each meeting-point of four squares, allow diagonal connections between black squares with probability p; otherwise allow diagonal connections between white squares. We show that there is a critical surface of pairs (λ, p), containing the pair $(1, 0.5)$, such that for (λ, p) lying above (respectively, below) the critical surface the black (resp. white) phase percolates, and on the critical surface neither phase percolates.

Key words: Dependent percolation, random sequential adsorption, critical surface
MSC: 60K35, 82B43

1 Introduction

Random sequential adsorption (abbreviated RSA throughout this paper) is a term for a family of probability models for irreversible particle deposition. Particles arrive at random locations and times onto a surface, and if accepted a particle blocks nearby locations on the surface from accepting future arrivals. Such models are of physical interest, as a modal for coating of a surface; see for example [6, 14]. We consider a discrete version of RSA on the initially empty integer lattice $\mathbb{Z}^2$, with the arrival time at a lattice site x given by an exponential random variable T_x with parameter λ_x, with $(T_x)_{x \in \mathbb{Z}^2}$ independent. All sites are either *empty*, *occupied* or *blocked*; an arrival at an empty site x causes it to become permanently occupied and all adjacent sites (that is, sites y such that $|x - y| = 1$ where $|\cdot|$ denotes the Euclidean norm) to become permanently blocked. If $\sup_x \lambda_x < \infty$ this model is well defined; see [11]. On this lattice we define the even (respectively, odd) sites to be those at an even (respectively, odd) graph distance from the origin.

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Ultimately, each site will be either occupied or blocked. The distribution of the occupied and blocked sites in this ultimate state is called the jamming distribution; under the jamming distribution the sites of $\mathbb{Z}^2$ are divided into an even phase and an odd phase, where the even phase consists of occupied even sites and blocked odd sites. Site percolation of the even phase was considered in [12], in the case where for some $\lambda > 0$ we have $\lambda_x = 1$ for odd x and $\lambda_x = \lambda$ for even x. The even phase is *monotone* in λ; that is, for $0 \leq \lambda < \lambda'$ there exists coupled realisations of the process just described with parameter λ and with parameter λ', such that the even phase for parameter λ is contained in the even phase for parameter λ'.

Penrose and Rosoman [12] proved that the critical parameter λ for RSA on the integer lattice $\mathbb{Z}^2$ is strictly greater than 1. The proof of this uses an *enhanced RSA* (denoted eRSA below) model on a new lattice called Λ throughout this paper. We associate with each site $x \in \mathbb{Z}^2$ a site $x' := x + (1/2, 1/2)$. The lattice Λ has vertex set $\cup_{x \in \mathbb{Z}^2}\{x, x'\}$, with an edge between sites $x \in \mathbb{Z}^2$ and $y \in \mathbb{Z}^2$ if $|x - y| = 1$, and an edge between x' and y if $|x' - y| = \frac{\sqrt{2}}{2}$ (here $|\cdot|$ is the Euclidean distance). We refer to the added sites x' as diamond sites, and the original sites x as octagon sites; each octagon site has degree 8 and each diamond site has degree 4 (see Fig 1).
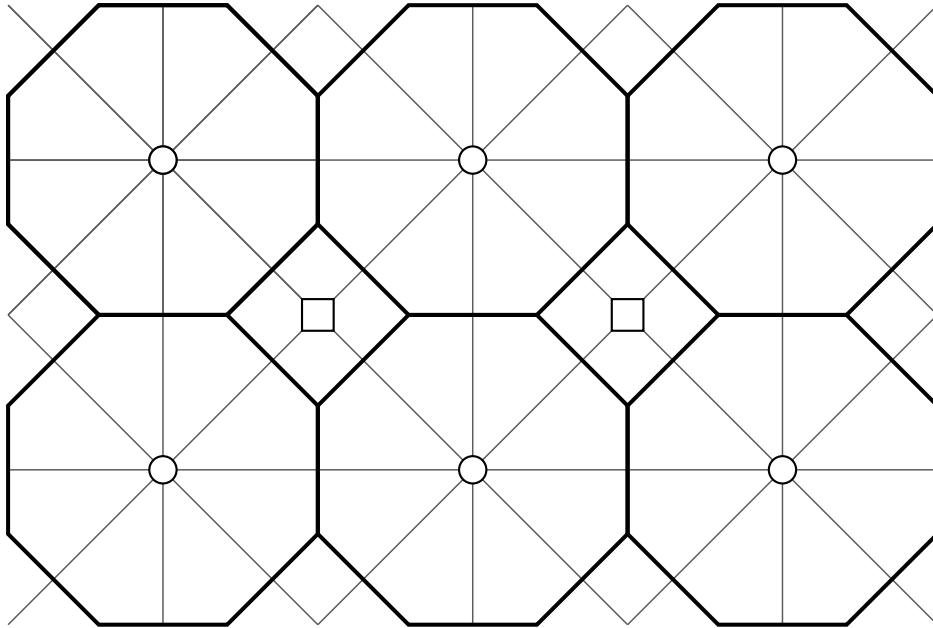

Figure 1: A section of the lattice used for enhanced RSA and the associated tiling of $\mathbb{R}^2$. The small circles represent octagon sites and the small squares are diamond sites.

We introduce an *enhancement parameter* $p \in [0, 1]$. Each of the diamond sites x' is independently taken to be in the even phase with probability p, and otherwise in the odd phase. Considering percolation on this new lattice, where a site is considered *black* if it is in the even phase and otherwise *white*, we say the even (resp. odd) phase *percolates* if there is an infinite component of black (resp. white) sites.

Taking the special case with $p = 1$ amounts to always allowing diagonal connections between black sites of $\mathbb{Z}^2$, and never allowing diagonal connections between white sites. Taking $p = 0$ amounts to the opposite. Considering the enhanced model enables us to

interpolate continuously between these extremes. Moroever, this model enjoys a duality relation whereby the even phase for parameters (λ, p) has the same distribution as the odd phase for parameters $(1/\lambda, 1 - p)$ (see Lemma 2.3 below).

In this paper we consider the enhanced model in its own right, with a further parameter $\lambda \in \mathbb{R}_+ := (0, \infty)$ and with $\lambda_x = 1$ for odd x and $\lambda_x = \lambda$ for even x. For $p \in [0, 1]$ we define the critical values

$$\lambda_c^+(p) := \inf\{\lambda : \text{ the even phase percolates in eRSA}(\lambda, p)\};$$
$$\lambda_c^-(p) := \sup\{\lambda : \text{ the odd phase percolates in eRSA}(\lambda, p)\}.$$

It is natural to ask whether these values coincide, and if so, to try to understand the behaviour of the critical surface in (λ, p)-space for this model; for example, the symmetry suggests that the pair $(\lambda = 1, p = 1/2)$ should be critical. Our main result provides some information on these issues.

Theorem 1.1. *(i) For each $p \in [0, 1]$ we have $\lambda_c^-(p) \leq \lambda_c^+(p)$, with equality whenever $0 < p < 1$.*

(ii) For each $p \in [0, 1]$ there is no percolation of the even phase for eRSA with parameters $(\lambda_c^+(p), p)$, and no percolation of the odd phase for eRSA with parameters $(\lambda_c^-(p), p)$.

(iii) It is the case that $\lambda_c^+(1/2) = 1$.

(iv) For any $\varepsilon \in (0, 1/2)$, the functions $\lambda_c^+ : [\varepsilon, 1] \to \mathbb{R}_+$ and $\lambda_c^- : [0, 1 - \varepsilon] \to \mathbb{R}_+$ are strictly decreasing and Lipschitz, and the inverse of the function $\lambda_c^+ : [\varepsilon, 1 - \varepsilon] \to [\lambda_c^+(1 - \varepsilon), \lambda_c^+(\varepsilon)]$ is also strictly decreasing and Lipschitz.

We conjecture that $\lambda_c^+(p) = \lambda_c^-(p)$ for all $p \in [0, 1]$ but we prove this only for $p \in (0, 1)$. It is clear from the theorem that the inverse function of $\lambda_c^+(\cdot)$ is the function $p_c^+(\cdot)$ defined by

$$p_c^+(\lambda) := \inf\{p : \text{ the even phase percolates in eRSA}(\lambda, p)\}.$$

An outline of the proof will be provided in Section 2 with details filled in in subsequent sections. Most of the work goes into showing that if the *odd* phase does *not* percolate at a certain (λ, p) (Assumption A), then after an arbitrarily small increase in either λ or p the *even* phase *does* percolate (Conclusion B). The strategy to prove this goes as follows. Under Assumption A, we shall adapt known methods to deduce that the even phase crosses an arbitrarily large rectangle of aspect ratio 3 the long way, with non-vanishing probability. Then using a suitable *sharp thresholds* result for increasing events in a finite product space (presented in Section 3, and perhaps of independent interest), we shall deduce in Proposition 2.2 that after increasing λ or p we have a crossing of such a rectangle with probability close to 1, and then a standard comparison with 1-dependent percolation yields Conclusion B.

To use our sharp thresholds result, we shall discretize time. We shall demonstrate that the approximation error involved in the discretization can be compensated for with a slight increase in the parameter λ or p that vanishes as the size of the rectangle approaches infinity. We do this using the method of *essential enhancements* (see for example [1]) to show that the effect of the discretization parameter is comparable to that of the enhancement parameter p.

Our strategy outlined above is related to to a method used by Bollobás and Riordan in [3] to prove that the critical value for Voronoi percolation in the plane is $1/2$, but is distinguished by our use of essential-enhancement techniques in the last step rather than the coupling construction appearing at a comparable stage in [3]. This method might be of use elsewhere. Indeed, we believe that these methods are likely to be relevant to showing similar results on existence of a sharply defined and smooth critical surface for other percolation models having two or more parameters and long-range dependence, provided correlations are sufficiently rapidly decaying (in the present instance this holds because of Lemma 2.1 below), and identifying actual critical values for such models with sufficient symmetry.

For example, consider random sequential deposition of monomers (at rate 1) and dimers (at rate α) onto the vertices of the triangular lattice with each monomer accepted if it arrives at a previously unoccupied site, and each dimer accepted if it arrives at a previously unoccupied pair of neighbouring sites. Suppose each monomer (respectively dimer) is black with probability $p \in [0, 1]$ (respectively $q \in [0, 1]$). Ultimately all sites will be occupied, and for fixed α we would expect that our methods could be adapted to show that there is a smooth critical surface in (p, q)-space passing through $(1/2, 1/2)$.

As another example, consider sequential deposition of hard and soft particles, where the hard particles exclude each other in RSA fashion, and any point not occupied by a hard particle acquires the colour of the first soft particle to arrive covering it; suppose hard particles are black with probability p and soft particles are black with probability q. This could be considered, with deposition either on the vertices of the triangular lattice with hard particles excluding each other from neighbouring sites, or in the continuum $\mathbb{R}^2$ with the particles given by unit disks (or some other shape). Again, it may be possible to adapt our methods to these models.

The continuum version of the last model without the hard particles (and therefore with finite range dependences) amounts to the so-called 'confetti percolation' or 'dead leaves' model, for which similar questions have been considered in [8] and [10]. In the latter paper, Müller deploys a different sharp threshold type result with weaker symmetry requirements; it would be very interesting to explore the possible application of those ideas in models such as those mentioned above.

2 Proof of Theorem 1.1

In this section we prove our theorem, but with the proof of certain key steps deferred to later sections. We first assemble some facts based adapting known methods to eRSA.

For $x, y \in \mathbb{Z}^2$, we shall say that the site x *affects* the site y if there is some self-avoiding path in $\mathbb{Z}^2$ starting at a neighbour of x (note: not at x itself) and ending at y, such that if the odd sites along this path are listed in order as $x_1, x_2, \ldots, x_m$, then $T_{x_1} \leq T_{x_2} \leq \cdots \leq T_{x_m}$.

Lemma 2.1. *Let $x, y \in \mathbb{Z}^2$ with $x \neq y$. With probability 1, if x does not affect y, then no change to the arrival time at x with all other arrival times remaining fixed can alter the state of y. Moreover, if $d(x, y)$ denotes the graph distance between x and y in $\mathbb{Z}^2$, then*

$$\mathbb{P}[x \text{ affects } y] \leq \frac{4^{d(x,y)}}{\lfloor d(x,y)/2 \rfloor !}. \tag{2.1}$$

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Proof. First note that (2.1) follows easily by the union bound.

Partition the sites of $\mathbb{Z}^2$ into *generations* $G_0, G_1, G_2, \ldots$, defined as follows. Set $G_0 = \{x\}$. Inductively, suppose for some k that $G_0, G_1, \ldots, G_k$ have been defined. For each $z \in \mathbb{Z}^2 \setminus \cup_{i=0}^k G_i$ we put $z \in G_{k+1}$ if and only if $T_z < T_w$ for all $w \in \mathbb{Z}^2 \setminus \cup_{i=1}^k G_i$ with w neighbouring z. Using (2.1) and the first Borel-Cantelli lemma, one can show that with probability 1, the sets $G_0, G_1, G_2, \ldots$ do indeed partition $\mathbb{Z}^2$.

We now prove the first assertion of the lemma, by induction on the generation containing y. If $y \in G_1$ and x does not affect y, then x is not a neighbour of y so all neighbours of y have later arrival times than T_y, and therefore y becomes occupied rather than blocked regardless of the arrival time at x.

For the inductive step, suppose for some $k \in \mathbb{N}$ that the assertion of the lemma holds for all $y \in \cup_{i=1}^k G_i$. Suppose that $z \in G_{k+1}$, and that x does not affect z. Enumerate the neighbouring sites of z with earlier arrival times than z as $y_1, \ldots, y_j$. These sites must all lie in $\cup_{i=0}^k G_i$ and moreover are not affected by x (else z would also be affected by x). Hence by the inductive hypothesis, the occupied/blocked status of sites $y_1, \ldots, y_j$ is not affected by any change to the arrival time at x, and hence the status of site z is also not affected by such a change (z is occupied if all of $y_1, \ldots, y_j$ are blocked). This completes the induction. $\square$

Let $\mathbb{P}_{\lambda,p}$ denote the probability measure associated with the enhanced RSA model where λ is the rate of arrivals at even sites and p is the enhancement parameter. We next provide the *Harris-FKG* inequality for this model. For any two black/white colourings α, β of the vertices of Λ (i.e., of the faces of the tiling), let us write $\alpha \prec \beta$ if the set of black sites in α is contained in the set of black sites in β. Let us say that an event E, defined in terms of the colouring induced by the RSA model, is *black-increasing* if it has the following property: for any two colourings α, β with $\alpha \prec \beta$, if $\alpha \in E$ then $\beta \in E$. For example, $H_{n,\rho}$ is black-increasing. Similarly, we say E is *white-increasing* if for any two colourings α, β with $\beta \prec \alpha$, if $\alpha \in E$ then $\beta \in E$.

Lemma 2.2. [Harris-FKG inequality.] *Let $\lambda > 0, p \in [0,1]$. If E and F are both black-increasing events or are both white-increasing events then $\mathbb{P}_{\lambda,p}(E \cap F) \geq \mathbb{P}_{\lambda,p}(E)\mathbb{P}_{\lambda,p}(F)$.*

Proof. The Harris-FKG inequality for RSA is given in section 5 of [13] and may then be deduced in the enhanced RSA model using the independence of the enhancement variables. $\square$

We shall refer to the following lemma as a *duality* relation.

Lemma 2.3. *Let $\lambda > 0, p \in [0,1]$. Then the even phase of eRSA with parameters (λ, p) percolates, if and only if the odd phase of of eRSA with parameters $(1/\lambda, 1-p)$ percolates.*

Proof. Consider first the eRSA process with parameters (λ, p). Now re-scale time by multiplying all arrival times by a factor of λ; the rescaled arrival times are exponential with rate 1 at even sites and rate $1/\lambda$ at odd sites. If we then also interchange the colours, then the new set of black sites is a realization of eRSA with parameters $(1/\lambda, 1-p)$. $\square$

Given a rectangle $R = [a,b] \times [c,d]$, and $r > 0$, define $E_{\text{dense}}(R, r)$ to be the event that no site in R is affected by any site outside $[a - r, b + r] \times [c - r, d + r]$. Using (2.1), one can readily prove the following which has appeared previously as Lemma 3.3 of [12].

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Lemma 2.4. *Let $\lambda > 0$, $\rho \geq 1$. Given $s > 0$, let $R_s = [1, \lfloor s \rfloor] \times [1, \lfloor \rho s \rfloor]$. Then $\mathbb{P}_\lambda[E_{\mathrm{dense}}(R_s, 2\lfloor s^{1/2} \rfloor)] \to 1$ as $s \to \infty$. Moreover, $E_{\mathrm{dense}}(R, r)$ depends only on the arrival times within the larger rectangle.*

We now discuss certain box crossings. We construct a dependent face percolation model on a truncated square tiling (shown by the darker lines in figure 1) as follows: colour the octagon centred at $x \in \mathbb{Z}^2$ black if x is in the even phase, otherwise colouring it white. As in Section 1 we denote the diamond at the top right corner of the octagon centred at x by x', and colour it black if the x' is in the even phase, and white otherwise. Given $\rho \in (0, \infty)$ and $n \in \mathbb{N}$ with $\rho n \geq 1$, let $H_{n,\rho}$ denote the event that there is a horizontal black crossing of the rectangle

$$R(2n, \rho) := [-\lfloor \rho n \rfloor, \lfloor \rho n \rfloor - 1] \times [-n, n - 1]$$

and set

$$h_\rho(n, \lambda, p) := \mathbb{P}_{\lambda, p}(H_{n, \rho}).$$

Also, define $h'_\rho(n, \lambda, p)$ similarly but in terms of a white crossing. That is, $h'_\rho(n, \lambda, p)$ denotes the probability that there is a horizontal white crossing of an arbitrary fixed $2\lfloor \rho n \rfloor$ by $2n$ rectangle for eRSA with parameters λ and p. Note that any given rectangle possesses either a horizontal black crossing or vertical white crossing.

Lemma 2.5. *There exist constants $\kappa > 0$, and $n_0 \in \mathbb{N}$, such that the even phase percolates if there exists $n \geq n_0$ with $h_3(n, \lambda, p) > 1 - \kappa$.*

Proof. This can be proved by a similar method to Theorem 1.1 of [3], namely comparison with 1-dependent bond percolation along with use of Lemma 2.4. $\square$

The next ingredient is an RSW type result relating the probability of crossing a large box of one (fixed) aspect ratio, to the probability of crossing a box of a different aspect ratio.

Lemma 2.6. *Let $\lambda > 0$, $p \in [0, 1]$, $\rho > 0$ be fixed. If $\limsup_{n \to \infty} h_\rho(n, \lambda, p) > 0$ then $\limsup_{n \to \infty} h_{\rho'}(n, \lambda, p) > 0$ for all $\rho' > 0$. If $\limsup_{n \to \infty} h'_\rho(n, \lambda, p) > 0$ then $\limsup_{n \to \infty} h'_{\rho'}(n, \lambda, p) > 0$ for all $\rho' > 0$.*

Proof. A weaker version of this result (with $\liminf$ rather than $\limsup$ in the hypothesis, and with $\rho = 1$) is given by the proof of Proposition 3.2 in [12], based on that of Theorem 4.1 of [3]. The details on how to convert the proof to the stronger statement given here can be found in [2, Section 4]. The argument uses the rapid decay of correlations (which follows from Lemma 2.1), the Harris-FKG inequality, and the invariance of the model under 90 degree rotations and under reflections in the y-axis. $\square$

Note that by symmetry, $h_1(n, 1, 1/2) = 1/2$, and therefore we have for all $\rho' > 0$ that $\limsup_{n \to \infty} h_{\rho'}(n, 1, 1/2) > 0$. Using the method of proof of the recent result in [15], it should be possible to show that in fact $\liminf_{n \to \infty} h_{\rho'}(n, 1, 1/2) > 0$, but we do not need this. On the other hand, in applying Lemma 2.6 we will need the case with $\rho = 1/3$ as well as the case with $\rho = 1$.

Lemma 2.7. *For eRSA with parameters (λ, p), the even phase percolates if and only if $\lim_{n \to \infty} h_3(n, \lambda, p) = 1$; the odd phase percolates if and only if $\lim_{n \to \infty} h_{1/3}(n, \lambda, p) = 0$.*

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Proof. By Lemma 2.5, it is immediate that $\lim_{n\to\infty} h_3(n, \lambda, p) = 1$ implies percolation of the even phase.

Now suppose that $\liminf_{n\to\infty} h_3(n, \lambda, p) < 1$; then $\limsup_{n\to\infty} h'_{1/3}(n, \lambda, p) > 0$. By Lemma 2.6, hence $\limsup_{n\to\infty} h'_3(n, \lambda, p) > 0$. We can thus find a sequence $(n_i)_{i\in\mathbb{N}}$ with $n_{i+1} > 4n_i$ such that $\liminf h'_3(n_i, \lambda, p) > 0$. For $i \in \mathbb{N}$ define rectangles $R_{i,j}, 1 \leq j \leq 4$, by

$$
\begin{aligned}
R_{i,1} &= [-3n_i, 3n_i] \times [-3n_i, -n_i] \\
R_{i,2} &= [n_i, 3n_i] \times [-3n_i, 3n_i] \\
R_{i,3} &= [-3n_i, 3n_i] \times [n_i, 3n_i] \\
R_{i,4} &= [-3n_i, n_i] \times [-3n_i, -3n_i].
\end{aligned}
$$

Let E_i denote the event that all four rectangles $R_{i,j}$ contain a long way crossing in the odd phase and that $\cap_{j=1}^4 E_{\text{dense}}(R_{i,j}, n_i/8)$ holds. By Lemma 2.4, $\mathbb{P}_{\lambda,p}(E_{\text{dense}}(R_{i,j}, n_i/8)) \to 1$ as $i \to \infty$, and hence by the Harris-FKG inequality, $\liminf \mathbb{P}_{\lambda,p}(E_i) > 0$. Since the events $(E_i)_{i\in\mathbb{N}}$ are independent, it follows that almost surely at least one of them occurs, and hence the cluster containing the origin in the even phase is almost surely finite.

The last part is proved similarly. $\square$

The next two propositions are key ingredients in the proof of Theorem 1.1; we defer their proof to Sections 5 and 6. The first of these says that the effect of a small change in λ on box crossing probabilities is comparable to that of a small change in p.

Proposition 2.1. *Let $\varepsilon \in (0, 1/2)$. Then there is a constant $c_1 = c_1(\varepsilon) \in (0, \infty)$ such that for any $(n, \lambda, p) \in \mathbb{N} \times [\varepsilon, 1/\varepsilon] \times [\varepsilon, 1 - \varepsilon]$ we have*

$$
c_1^{-1} \frac{\partial h_3(n, \lambda, p)}{\partial \lambda} \leq \frac{\partial h_3(n, \lambda, p)}{\partial p} \leq c_1 \frac{\partial h_3(n, \lambda, p)}{\partial \lambda}, \tag{2.2}
$$

and moreover the second inequality of (2.2) holds for any $(n, \lambda, p) \in \mathbb{N} \times [\varepsilon, 1/\varepsilon] \times [0, 1]$.

The last key ingredient says that if (at some (λ, p)) we have non-vanishing probability of crossing a large rectangle of fixed aspect ratio, then after a slight increase of either λ or p we have probability close to 1 of crossing a rectangle of aspect ratio 3 the long way.

Proposition 2.2. *Let $\lambda > 0$, $p \in (0, 1)$ and $\varepsilon > 0$ with $p + \varepsilon < 1$. Suppose for some $\rho > 0$ that $\limsup_{n\to\infty} h_\rho(n, \lambda, p) > 0$. Then*

$$
\limsup_{n\to\infty} h_3(n, \lambda + \varepsilon, p) = 1 \tag{2.3}
$$

and

$$
\limsup_{n\to\infty} h_3(n, \lambda, p + \varepsilon) = 1. \tag{2.4}
$$

We can now prove of Theorem 1.1, using the strategy outlined in Section 1.

Proof of Theorem 1.1. Let $n_0 \in \mathbb{N}$ and $\kappa > 0$ be as in Lemma 2.5. Let S denote the set of (λ, p) such that $h_3(n, \lambda, p) > 1 - \kappa$ for some $n \geq n_0$. Since $h_3(n, \lambda, p)$ is continuous in λ and p for any fixed n, the set S is open in $(0, \infty) \times [0, 1]$, and the even phase percolates

for any $(\lambda, p) \in S$. Also, if $(\lambda, p) \notin S$ then $\limsup(h_3(n, \lambda, p)) \leq 1 - \kappa < 1$, and thus there is no percolation by Lemma 2.7. Hence, S is the set of (λ, p) for which the even phase percolates.

Similarly, with S' denoting the set of values (λ, p) for which the odd phase percolates, the set S' is also open in $(0, \infty) \times [0, 1]$. Also $S \cap S' = \emptyset$ by Lemma 2.7. Since $\lambda_c^+(p) = \inf\{\lambda : (\lambda, p) \in S\}$, and $\lambda_c^-(p) = \sup\{\lambda : (\lambda, p) \in S'\}$, this gives us the inequality $\lambda_c^-(p) \leq \lambda_c^+(p)$.

For $(\lambda, p), (\lambda', p') \in (0, \infty) \times [0, 1]$, let us write $(\lambda', p') \succ (\lambda, p)$ to mean that $\lambda \leq \lambda'$ and $p \leq p'$ with at least one of these inequalities being strict.

Suppose $(\lambda, p) \notin S'$ and $0 < p < 1$. Then $\limsup_{n \to \infty} h_{1/3}(n, \lambda, p) > 0$ by Lemma 2.7. Hence by Proposition 2.2, for any $(\lambda', p') \succ (\lambda, p)$ we have $\limsup_{n \to \infty} h_3(n, \lambda', p') = 1$. Therefore $(\lambda', p') \in S$. In other words, for $0 < p < 1$ we have

$$(\lambda, p) \notin S' \implies (\lambda', p') \in S \quad \forall (\lambda', p') \succ (\lambda, p). \tag{2.5}$$

Hence for $0 < p < 1$ we have $\lambda_c^-(p) = \lambda_c^+(p)$. Thus we have part (i) of our theorem, and part (ii) follows from the fact that the sets S and S' are open.

Since $h_1(n, 1, 1/2) = 1/2$ for all n, we have that $\limsup h_3(n, 1, 1/2) < 1$, so by Lemma 2.7 we have $(1, 1/2) \notin S$. Hence by duality, also $(1, 1/2) \notin S'$, and part (iii) follows.

For part (iv), the strict monotonicity of $\lambda_c(\cdot)$ follows from (2.5) and the fact that S is open. We next prove the Lipschitz continuity of $\lambda_c(\cdot)$.

Let $\varepsilon \in (0, 1/2)$. By Theorem 2.1 of [12], $\lambda_c^+(0) < 10$, and hence by duality, $\lambda_c^-(1) > 0.1$. By Proposition 2.1, we can find $c_1 \in (1, \infty)$ such that for $(\lambda, p, n) \in [0.1, 11] \times [0, 1] \times \mathbb{N}$, we have the second inequality of (2.2).

Let $p \in [\varepsilon, 1]$. Then for any $\lambda \in (\lambda_c^+(p), 10)$, we have $\lambda \in S$ so we can find $n \geq n_0$ such that $h_3(n, \lambda, p) \geq 1 - \kappa$. Then by the second inequality of (2.2), for such n and for $0 < \delta < \varepsilon/c_1$ we have

$$h_3(n, \lambda + c_1 \delta, p - \delta) \geq h_3(n, \lambda, p) \geq 1 - \kappa$$

so that $(\lambda + c_1 \delta, p - \delta) \in S$ and hence $\lambda_c^+(p - \delta) \leq \lambda_c^+(p) + c_1 \delta$. This gives the Lipschitz continuity of $\lambda_c^+(\cdot)$ on $[\varepsilon, 1]$.

Now suppose $0 \leq p \leq 1 - \varepsilon$. By duality (Lemma 2.3) we have $\lambda_c^-(p) = 1/\lambda_c^+(1 - p)$. Thus for $0 < \delta < \min(\varepsilon, (100 c_1)^{-1})$, using that $\lambda_c^+(1 - p) < 10$ we also have

$$\lambda_c^+(1 - p - \delta) \leq \lambda_c^+(1 - p) + c_1 \delta \leq \frac{1}{\lambda_c^-(p) - c' \delta},$$

for $c' = 100 c_1$. Hence by duality again, $\lambda_c^-(p + \delta) \geq \lambda_c^-(p) - c' \delta$. This shows the Lipschitz continuity of $\lambda_c^-(\cdot)$ on $[0, 1 - \varepsilon]$.

For $\lambda_c^+(1) < \lambda < \lambda_c^-(0)$, set $p_c^+(\lambda) := \inf\{p : (\lambda, p) \in S\}$. By (2.5) and the fact that S is open, the function $p_c^+(\cdot)$ is strictly decreasing. By a similar argument to the above (now using the first inequality of (2.2)), we may show the Lipschitz continuity of $p_c^+(\lambda)$ as a function of λ for $\lambda_c^+(1) + \varepsilon \leq \lambda \leq \lambda_c^-(0) - \varepsilon$. Thus the restriction of the function $\lambda_c^+(\cdot)$ to the domain $[1 - \varepsilon, \varepsilon]$ has a Lipschitz inverse, namely $p_c^+(\cdot)$. $\qquad\square$

So far, we have used only the generic properties mentioned in the proof of Lemma 2.6, along with duality. For the enhancement estimates required below to prove Propositions 2.1 and 2.2, we shall require arguments more specific to this particular model.

3 A sharp thresholds result

The *sharp threshold property* [7] for increasing events in $\{0,1\}^n$ says that for any such event and any fixed $\eta \in (0, 1/2)$, when n is large the threshold value of p above which the probability of such an event (under product measure with parameter p) exceeds $1 - \eta$, is only slightly larger than the corresponding threshold for the event to have probability at least η.

In Proposition 3.1 below, we present a similar threshold result for events in $\{0, 1, \ldots, k\}^n$ for any fixed k, satisfying a symmetry assumption. Such a result was given in [4] for the case $k = 2$; we adapt this to general k and give a more detailed proof than that of [4]. Later, we shall use Proposition 3.1 to prove Proposition 2.2.

Let $k \in \mathbb{N}$. For $n, m \in \mathbb{N}$, a subset $E \subset \{0, 1, ..., k\}^n$ is said to have symmetry of order m if there is a group action on $[n] := \{1, 2, ..., n\}$ in which each orbit has size at least m, such that the induced action on $\{0, 1, ..., k\}^n$ preserves E; for instance, if n is even then a subset $E \subset \{0, 1, ..., k\}^{n^2}$ which is preserved by even translations of the n by n torus $[n] \times [n]$ (identified with $[n^2]$) would have symmetry of order $n^2/2$.

Given a probability vector $\boldsymbol{p} = (p_0, p_1, ..., p_k)$ (i.e., a finite vector with nonnegative entries summing to 1), we write $\mathbb{P}_{\boldsymbol{p}}$ for the probability measure on $\{0, \ldots, k\}$ with probability mass function $\boldsymbol{p}$, and for $n \in \mathbb{N}$ we write $\mathbb{P}_{\boldsymbol{p}}^n$ for the n-fold product of this probability measure (a probability measure on $\{0, \ldots, k\}^n$). We say that $E \subset \{0, 1, \ldots, k\}^n$ is *increasing*, if for every $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ in $\{0, 1, \ldots, k\}^n$ such that $x \in E$ and $y_i \geq x_i$ for $i \in \{0, 1, \ldots, n\}$, we have $y \in E$.

Given probability vectors $\boldsymbol{p} = (p_0, \ldots, p_k)$ and $\boldsymbol{q} = (q_0, \ldots, q_k)$, we say that $\boldsymbol{q}$ *dominates* $\boldsymbol{p}$ if for $j = 0, 1, 2, \ldots, k - 1$ we have $\sum_{i=0}^{j}(p_i - q_i) \geq 0$. Note that $\boldsymbol{q}$ dominates $\boldsymbol{p}$, if and only if there are coupled random variables X, Y taking values in $\{0, 1, \ldots, k\}$ such that X has distribution $\mathbb{P}_{\boldsymbol{p}}$ and Y has distribution $\mathbb{P}_{\boldsymbol{q}}$ and $Y \geq X$ almost surely. Thus, if $\boldsymbol{q}$ dominates $\boldsymbol{p}$ then $\mathbb{P}_{\boldsymbol{q}}^n(E) \geq \mathbb{P}_{\boldsymbol{p}}^n(E)$ for any $n \in \mathbb{N}$ and any increasing $E \subset \{0, 1, \ldots, k\}^n$.

Proposition 3.1. *Let* $k, n, m \in \mathbb{N}$, *let* $\eta \in (0, 1/2)$ *and let* $\gamma > 0$. *Suppose* $\boldsymbol{p} = (p_0, p_1, \ldots, p_k)$ *and* $\boldsymbol{q} = (q_0, q_1, \ldots, q_k)$ *are probability vectors such that* $p_0 \geq \gamma$, $p_k \leq 1 - \gamma$ *and* $\boldsymbol{q}$ *dominates* $\boldsymbol{p} + (-\gamma, 0, \ldots, 0, \gamma)$. *Let* $q_{\max}$ *denote the second largest of the numbers* $p_0, \ldots, p_{k-1}, p_k + \gamma$, *and suppose also that*

$$\gamma \log m \geq 200k^2 \log(1/\eta) q_{\max} \log(4/q_{\max}). \tag{3.1}$$

Then for any increasing $E \subset \{0, 1, \ldots, k\}^n$ *with symmetry of order* m, *and with* $\mathbb{P}_{\boldsymbol{p}}^n(E) > \eta$, *we have* $\mathbb{P}_{\boldsymbol{q}}^n(E) > 1 - \eta$.

The remainder of this section is devoted to proving this, via a series of lemmas.

Given a probability vector $\boldsymbol{p} = (p_0, p_1, ..., p_k)$, define

$$\beta_{\boldsymbol{p}}(x) \quad := \quad \max\left\{ j \in \{0, \ldots, k\} : \sum_{i=0}^{j-1} p_i \leq x \right\}, \qquad x \in [0, 1).$$

where by definition we set $\sum_{i=0}^{-1} p_i = 0$. Define $p_{\max}(\boldsymbol{p})$ to be the *second largest* of the numbers $p_0, p_1, \ldots, p_k$. Given $\ell \in \mathbb{N}$, let $h_\ell : [0, 1) \to [0, 1)$ be the function which inverts the ℓth digit of the binary expansion of a number (using the terminating expansion

wherever there is a choice). Now we let U be a uniform $(0,1)$ distributed random variable, and for $f : \{0, 1, \ldots, k\} \to \{0, 1\}$ define

$$w_{\ell,\boldsymbol{p}}(f) = \mathbb{P}\left[f \circ \beta_{\boldsymbol{p}}(U) \neq f \circ \beta_{\boldsymbol{p}}(h_\ell(U))\right], \quad \ell \in \mathbb{N};$$

$$w_{\boldsymbol{p}}(f) = \sum_{\ell=1}^{\infty} w_{\ell,\boldsymbol{p}}(f).$$

Lemma 3.1. *Let $k \in \mathbb{N}$. Then for any probability vector $\boldsymbol{p} = (p_0, p_1, \ldots, p_k)$ with $p_i > 0$ for all i, and any function $f : \{0, 1, \ldots, k\} \to \{0, 1\}$, we have*

$$w_{\boldsymbol{p}}(f) \leq 3k^2 p_{\max}(\boldsymbol{p}) \log(4/p_{\max}(\boldsymbol{p})). \tag{3.2}$$

Proof. For $0 \leq j \leq k-1$ define $q_j := \sum_{i=0}^{j} p_i$ and $q_j^* := \min(q_j, 1 - q_j)$. Then for $\ell \in \mathbb{N}$,

$$w_{\ell,\boldsymbol{p}}(f) \leq 2 \sum_{j=0}^{k-1} \mathbb{P}(U < q_j < h_\ell(U)) \leq 2 \sum_{j=0}^{k-1} \min(q_j^*, 2^{-\ell}).$$

Hence,

$$w_{\boldsymbol{p}}(f) \leq \sum_{j=0}^{k-1} \left(\sum_{\ell=1}^{\lfloor \log_2(1/q_j^*) \rfloor} 2q_j^* + \sum_{\ell=\lceil \log_2(1/q_j^*) \rceil}^{\infty} 2^{1-\ell} \right)$$

$$\leq \sum_{j=0}^{k-1} \left(2q_j^* \log_2(1/q_j^*) + 4q_j^* \right) = \sum_{j=0}^{k-1} 2q_j^* \log_2(4/q_j^*).$$

By routine calculus $p \log_2(4/p)$ is increasing in p for $p \in (0, 1)$. Hence for all j in the sum,

$$q_j^* \log_2(4/q_j^*) \leq q_j \log_2(4/q_j) \leq \sum_{\ell=0}^{j} p_\ell \log_2(4/p_\ell), \tag{3.3}$$

and also

$$q_j^* \log_2(4/q_j^*) \leq (1 - q_j) \log_2(4/(1 - q_j)) \leq \sum_{\ell=j+1}^{k} p_\ell \log_2(4/p_\ell). \tag{3.4}$$

Choose $s \in \{0, 1, \ldots, k\}$ such that $p_s = \max(p_0, p_1, \ldots, p_k)$. Using (3.3) for $j < s$ and (3.4) for $j \geq s$, we obtain

$$w_{\boldsymbol{p}}(f) \leq k(k+1)p_{\max}(\boldsymbol{p}) \log_2(4/p_{\max}(\boldsymbol{p})).$$

Since $k(k+1) \leq 2k^2$ and $\log 2 > 2/3$, the result (3.2) follows. $\qquad\square$

Given $k, n \in \mathbb{N}$, given $f : \{0, 1, \ldots, k\}^n \to \{0, 1\}$ and $j \in \{0, \ldots, k\}$, and given $x = (x_1, \ldots, x_n) \in \{0, 1, \ldots, k\}^n$, we say the jth coordinate of x is *pivotal* for f if there exists $y = (y_1, \ldots, y_n) \in \{0, 1, \ldots, k\}^n$, with $y_i = x_i$ for all $i \neq j$, such that $f(x) \neq f(y)$. Given also a probability vector $\boldsymbol{p} = (p_0, \ldots, p_k)$ we define the *influence* $I_{f,\boldsymbol{p}}(j)$ of the jth coordinate on f as the probability that the jth coordinate of X is pivotal for f, where here X is a random element of $\{0, 1, \ldots, k\}^n$ with distribution $\mathbb{P}_{\boldsymbol{p}}^n$.

Lemma 3.2. *Let $k, n \in \mathbb{N}$. For any probability vector $\boldsymbol{p} = (p_0, p_1, \ldots, p_k)$ with all $p_i > 0$, any function $f : \{0, 1, \ldots, k\}^n \to \{0, 1\}$, any $q \in [p_{\max}(\boldsymbol{p}), 1]$ and any $a \in (0, 1/16]$, if*

$$I_{f,\boldsymbol{p}}(j) \leq aq^2 (\log(4/q))^2, \qquad \forall j \in [n], \tag{3.5}$$

then setting $t = \mathbb{P}_{\boldsymbol{p}}^n(f^{-1}(1)) = \mathbb{E}f(X)$, we have that

$$\sum_{j=1}^{n} I_{f,\boldsymbol{p}}(j) \geq \frac{t(1-t)\log(1/a)}{24k^2 q \log(4/q)}. \tag{3.6}$$

Given Lemma 3.1, the proof of Lemma 3.2 is similar to that of Lemma 2 of [4]. However, the argument given there (even in the ArXiv version, which has more detail than the published version) is quite sketchy, and 'not intended to be read on its own'; it relies on arguments from Theorems 3.1 and 3.4 of [7], and both of these papers rely heavily on arguments from [5], which is itself rather concise. Moreover, none of these papers is entirely free of minor errors, which does not aid readability. Therefore to make this presentation more self-contained, and also to give explicit constants in the bounds, we think it worthwhile to give a detailed proof. However, we defer it to the Appendix.

We now give the proof of Proposition 3.1, which is adapted from that of Lemma 1 in [4].

Proof of Proposition 3.1. Note that $\gamma \leq \min(p_0, p_k + \gamma) \leq q_{\max}$. Therefore by the assumption (3.1), $\log m \geq 200(\log 2) \log(4/q_{\max})$. Hence $m \geq q_{\max}^{-9}$, and also $m \geq 16^4$.

For $0 \leq h \leq \gamma$ set $\boldsymbol{r}(h) = \boldsymbol{p} + (-h, 0, \ldots, 0, h)$. Let $g(h) = \mathbb{P}_{\boldsymbol{r}(h)}^n(E)$. By assumption, $\boldsymbol{q}$ dominates $\boldsymbol{r}(\gamma)$. Therefore $\mathbb{P}_{\boldsymbol{q}}^n(E) \geq \mathbb{P}_{\boldsymbol{r}(\gamma)}^n(E) = g(\gamma)$. We shall use a form of the Margulis-Russo formula, namely

$$g'(h) = I_{f,\boldsymbol{r}(h)} := \sum_{j=1}^{n} I_{f,\boldsymbol{r}(h)}(j), \qquad \forall h \in (0, \gamma), \tag{3.7}$$

where f is the indicator of event E and $I_{f,\boldsymbol{p}}(j)$ is the influence of the jth coordinate on the function f, as in Lemma 3.2. To see (3.7), for $h_1, \ldots, h_n \in (0, \gamma)$ let $u(h_1, \ldots, h_n)$ denote the probability of event E under the measure $\prod_{i=1}^{n} \mathbb{P}_{\boldsymbol{r}(h_i)}$ and for probability vectors $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ on $\{0, 1 \ldots, k\}$ and $j \in [n]$ let $I_{f,(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)}(j)$ denote the probability that the jth coordinate of X is pivotal for f, where X is a random element of $\{0, 1 \ldots, k\}^n$ with distribution $\prod_{i=1}^{n} \boldsymbol{p}_i$. Then for $j \in [n]$ and $\varepsilon > 0$ with $h_j + \varepsilon < \gamma$, we can find coupled $\{0, 1, \ldots, k\}^n$-valued random vectors X and X' with respective distributions $\prod_{i=1}^{n} \mathbb{P}_{\boldsymbol{r}(h_i)}$ and $\prod_{i=1}^{n} \mathbb{P}_{\boldsymbol{r}_i'}$, where we set $\boldsymbol{r}_i' = \boldsymbol{r}(h_i)$ except for $i = j$, and $\boldsymbol{r}_j' = \boldsymbol{r}(h_j + \varepsilon)$, and such that $\mathbb{P}[X = X'] = 1 - \varepsilon$ and if $X \neq X'$ then $X_j = 0$ and $X_j' = k$, with $X_i = X_i'$ for all $i \neq j$. Then $f(X') \leq f(X)$, with equality except when (i) $X \neq X'$ and (ii) the jth coordinate of X is pivotal for f. Therefore

$$\mathbb{P}[f(X') \neq f(X)] = \varepsilon I_{f,(\boldsymbol{r}(h_1), \ldots, \boldsymbol{r}(h_k))}(j)$$

so that $\frac{\partial}{\partial h_j} u(h_1, \ldots, h_n) = I_{f,(\boldsymbol{r}(h_1), \ldots, \boldsymbol{r}(h_k))}$, and then we obtain (3.7) by the chain rule, since $g(h) = u(h, h, \ldots, h)$.

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Next we show that for $0 \leq h \leq \gamma$ we have

$$I_{f,\boldsymbol{r}(h)} \geq \frac{g(h)(1 - g(h)) \log m}{96 k^2 q_{\max} \log(4/q_{\max})}. \tag{3.8}$$

First suppose $I_{f,\boldsymbol{r}(h)}(j) \geq m^{-1/2}$ for some $j \in [n]$. Then by the symmetry assumption we have $I_{f,\boldsymbol{r}(h)}(j) \geq m^{-1/2}$ for at least m values of j, so that (using that $m \geq q_{\max}^{-9}$) we have $I_{f,\boldsymbol{r}(h)} \geq m^{1/2} \geq m^{1/3} q_{\max}^{-1.5}$, and since $\log m \leq 3 m^{1/3}$, this implies (3.8).

Now suppose instead that $I_{f,\boldsymbol{r}(h)}(j) < m^{-1/2}$ for all $j \in [n]$. Then since $m \geq q_{\max}^{-9}$ we have $I_{f,\boldsymbol{r}(h)}(j) < m^{-1/4} q_{\max}^2$ for all $j \in [n]$. Setting $a := \max_{j \in [n]} I_{f,\boldsymbol{r}(h)}(j)/(q_{\max}^2 (\log(4/q_{\max}))^2)$, we have that $a \leq m^{-1/4} \leq 1/16$; also $p_{\max}(\boldsymbol{r}(h)) \leq q_{\max}$, so by Lemma 3.2 we have

$$I_{f,\boldsymbol{r}(h)} \geq \frac{g(h)(1 - g(h)) \log(1/a)}{24 k^2 q_{\max} \log(4/q_{\max})} \geq \frac{g(h)(1 - g(h)) \log m}{96 k^2 q_{\max} \log(4/q_{\max})}$$

which implies (3.8).

For $0 \leq h \leq \gamma$ let $\tilde{g}(h) = \log(g(h)/(1 - g(h)))$. By (3.7) and (3.8) we have

$$\frac{d\tilde{g}}{dh} = (g(1-g))^{-1} \frac{dg}{dh} \geq \frac{\log m}{96 k^2 q_{\max} \log(4/q_{\max})}.$$

Since $g(0) = \mathbb{P}_{\boldsymbol{p}}^n(E) \geq \eta$ by assumption, we have $\tilde{g}(0) \geq \log \eta = -\log(1/\eta)$, and using the assumption (3.1), we obtain that

$$\tilde{g}(\gamma) \geq -\log(1/\eta) + \frac{\gamma \log m}{96 k^2 q_{\max} \log(4/q_{\max})} \geq \log(1/\eta)$$

which implies $g(\gamma) > 1 - \eta$, and therefore also $\mathbb{P}_{\boldsymbol{q}}^n(E) > 1 - \eta$. $\qquad\square$

4 Box crossings for eRSA

We now return to eRSA. As mentioned in Section 1, we shall apply Theorem 3.1 using a discretization of time. We shall compensate the error due to this, by introducing a time-*delay* at the even sites. In this section, we therefore consider a version of eRSA where the arrivals at the even sites are slightly delayed. We develop Margulis-Russo type formulae for the partial derivatives of box-crossing probabilities with respect to the parameters λ, p and the delay parameter, and estimates for the quantities arising from these formulae. We shall use these later to prove Propositions 2.1 and 2.2.

Given $\delta \geq 0$, we construct eRSA with arrivals at even sites delayed by δ from a collection of independent variables denoted T_x and $T_{x'}$, defined for $x \in \mathbb{Z}^2$. Here T_x is exponential with parameter 1 for odd x and with parameter λ for even x, while $T_{x'}$ is a uniform(0,1) random variable used to determine whether the diamond site x' is black or white. The arrival time t_x at x is $t_x = T_x$ for odd x and is $t_x = T_x + \delta$ for even x.

Since the precise arrival times at the sites do not matter for the resulting distribution, merely the order of arrivals, we have the same jamming distribution if we move the arrival times at all octagon sites forward by amount δ. Then by conditioning on the first arrival time at an odd site being at least δ and using the memoryless property of the exponential distribution we can arrive at the same distribution if the arrival time t_x at an even site is

T_x and at an odd site is 0 with probability $1-e^{-\delta}$, otherwise taking the value T_x. Therefore we now assume that as well as the variables T_x and $T_{x'}$, $x \in \mathbb{Z}^2$, we are provided with uniform(0,1) random variables $U_x, x \in \mathbb{Z}^2$. For $x \in \mathbb{Z}^2$ we now set the arrival time t_x to be 0 if x is odd and $U_x \le 1 - e^{-\delta}$; otherwise, we set t_x to be T_x. We set x' to be in the even phase if $T_{x'} < p$ and in the odd phase otherwise. Let $\mathbb{P}_{\lambda,p,\delta}$ denote the resulting jamming distribution.

Given $\rho \in (0, \infty)$ and $n \in \mathbb{N}$ with $\rho n \ge 1$, let $H_{n,\rho}$ denote the event that there is a horizontal black crossing of the rectangle

$$R(2n, \rho) := [-\lfloor \rho n \rfloor, \lfloor \rho n \rfloor - 1] \times [-n, n-1]$$

in the dependent face percolation model described in Section 2, and set

$$h_\rho(n, \lambda, p, \delta) := \mathbb{P}_{\lambda,p,\delta}(H_{n,\rho}).$$

Next we introduce the concept of a site being *pivotal* for the event $H_{n,\rho}$. The definition will depend on whether it is an odd site, an even site or a diamond site.

We shall say that an odd site x is pivotal for the event $H_{n,\rho}$ if $H_{n,\rho}$ occurs when we set the arrival time t_x to T_x but if we were to change the arrival time t_x to 0 (leaving all other variables constant), $H_{n,\rho}$ would no longer occur.

We shall say that an even site x is pivotal for the event $H_{n,\rho}$ if this event occurs when the arrival time at the site x is T_x, but does not occur if we delay the arrival time at x by an independent exponential random variable with rate λ called T.

For $y \in \mathbb{Z}^2$, we say that the diamond site y' is pivotal for the event $H_{n,\rho}$ if $H_{n,\rho}$ occurs when y' is black but does not occur when y' is white.

For any octagon or diamond site z we define

$$\phi_{\lambda,p,\delta,\rho}(n, z) := \mathbb{P}_{\lambda,p,\delta}[z \text{ is pivotal for event } H_{n,\rho}]$$

Proposition 4.1. *For any λ, n, p and ρ, and for any $\delta \ge 0$ it is the case that*

$$\frac{\partial h_\rho(n, \lambda, p, \delta)}{\partial p} = \sum_{x \in \mathbb{Z}^2} \phi_{\lambda,p,\delta,\rho}(n, x'), \tag{4.1}$$

$$\frac{\partial h_\rho(n, \lambda, p, \delta)}{\partial \lambda} = \lambda^{-1} \sum_{x \in \mathbb{Z}^2 : x \text{ even}} \phi_{\lambda,p,0,\rho}(n, x), \tag{4.2}$$

and

$$\frac{\partial h_\rho(n, \lambda, p, \delta)}{\partial \delta} = -e^{-\delta} \sum_{x \in \mathbb{Z}^2 : x \text{ odd}} \phi_{\lambda,p,\delta,\rho}(n, x), \tag{4.3}$$

where the partial derivative at $\delta = 0$ is interpreted as a one-sided right derivative.

Proof. Equations (4.1) and (4.2) are as in Proposition 4.1 of [12], and the proof there translates directly to this model.

For (4.3), fix n, p, λ, δ. Enumerate the odd sites of $\mathbb{Z}^2$ in some manner as $x_1, x_2, \dots$. Given $k \in \mathbb{N}$ and $\varepsilon > 0$, let $\mathbb{P}_{\delta,k,\delta+\varepsilon}$ denote the probability measure for a model with

enhancement parameter p and with arrival times $t_x, x \in \mathbb{Z}^2$, defined as follows. Let the variables $T_x, T_{x'}, U_x, x \in \mathbb{Z}^2$, be as before. Set $t_x = T_x$ for even x, and set

$$t_x = \begin{cases} 0 & \text{if } U_x < 1 - e^{-\delta} \\ T_x & \text{otherwise,} \end{cases} \qquad x \in \{x_1, \ldots, x_{k-1}\}$$

and

$$t_x = \begin{cases} 0 & \text{if } U_x < 1 - e^{-\delta-\varepsilon} \\ T_x & \text{otherwise,} \end{cases} \qquad x \in \{x_k, x_{k+1}, x_{k+2}, \ldots\}.$$

For $y, z \in \mathbb{Z}^2$ let the notion of 'y *even-affects* z' be defined in the same manner as 'y affects z', but in terms of the arrival times at the *even* sites in a path from y to z being in increasing order, rather than the odd sites. Let $A(x)$ be the event that the site x even-affects some site in $R(2n, \rho)$. Then

$$0 \le h_\rho(n, \lambda, p, \delta) - \mathbb{P}_{\delta,k,\delta+\varepsilon}[H_{n,\rho}] \le \mathbb{P}_{\delta,k,\delta+\varepsilon}[\cup_{j=k}^\infty A(x_j)]$$
$$\to 0 \text{ as } k \to \infty,$$

by (2.1) with the word 'affects' replaced by 'even-affects', which is applicable since the arrival rates at all *even* sites are the same. Thus,

$$h_\rho(n, \lambda, p, \delta + \varepsilon) - h_\rho(n, \lambda, p, \delta) = \mathbb{P}_{\delta,1,\delta+\varepsilon}[H_{n,\rho}] - \lim_{k\to\infty} \mathbb{P}_{\delta,k,\delta+\varepsilon}[H_{n,\rho}]$$
$$= \sum_{k=1}^\infty (\mathbb{P}_{\delta,k,\delta+\varepsilon}[H_{n,\rho}] - \mathbb{P}_{\delta,k+1,\delta+\varepsilon}[H_{n,\rho}]).$$

Given $\delta' > 0$ we now define $F_k(\delta, \delta')$ to be the event that $H_{n,\rho}$ occurs if we take the arrival time at site x_k to be T_{x_k}, but not if we take it to be 0, where an odd site x_j has arrival time 0 with probability $1 - e^{-\delta}$ if $j < k$ and with probability $1 - e^{-\delta'}$ if $j > k$, otherwise having as arrival time T_{x_j} (the dependence of F on p, λ and n is suppressed). With the variables as described above, we see that

$$\mathbb{P}_{\delta,k,\delta+\varepsilon}[H_{n,\rho}] - \mathbb{P}_{\delta,k+1,\delta+\varepsilon}[H_{n,\rho}] = -e^{-\delta}(1 - e^{-\varepsilon})\mathbb{P}[F_k(\delta, \delta+\varepsilon)]. \tag{4.4}$$

Couple $F_k(\delta, \delta + \varepsilon)$ and $F_k(\delta, \delta)$ by fixing the collection of random variables U_{x_j} for $j \in \mathbb{N}$. For $K \in \mathbb{N}$ let $B(2K + 1) := [-K, K] \times [-K, K]$. Then for any integer $K > 3n$ we see that

$$F_k(\delta, \delta + \varepsilon) \triangle F_k(\delta, \delta) \subset \left(\cup_{x \in \mathbb{Z}^2 \setminus B(2K+1)} A(x) \right)$$
$$\cup \left(\cup_{\{j > k : x_j \in B(2K+1)\}} \{1 - e^{-\delta} < U_{x_j} < 1 - e^{-(\delta+\varepsilon)}\} \right).$$

For any fixed K, the probability of the event

$$\cup_{\{j > k : x_j \in B(2K+1)\}} \{1 - e^{-\delta} < U_{x_j} < 1 - e^{-(\delta+\varepsilon)}\}$$

vanishes as $\varepsilon \downarrow 0$, and the probability of the event $\cup_{x \in \mathbb{Z}^2 \setminus B(2K+1)} A(x)$ is independent of ε and vanishes as $K \to \infty$. Then (4.4) yields

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \left(\mathbb{P}_{\delta,k,\delta+\varepsilon}[H_{n,\rho}] - \mathbb{P}_{\delta,k+1,\delta+\varepsilon}[H_{n,\rho}] \right) = -e^{-\delta}\mathbb{P}[F_k(\delta, \delta)] \tag{4.5}$$
$$= -e^{-\delta}\phi_{\lambda,p,\delta}(n, x_k).$$

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Finally, using Lemma 2.1, note that $\mathbb{P}[F_k(\delta, \delta + \varepsilon)]$ is bounded by $\mathbb{P}[A(x_k)]$, which is independent of ε and summable in k by (2.1), so by (4.4), (4.5), and the dominated convergence theorem we have

$$\frac{\partial^+ h_\rho}{\partial \delta} = \lim_{\varepsilon \downarrow 0} \frac{h_\rho(n, \lambda, p, \delta + \varepsilon) - h_\rho(n, \lambda, p, \delta)}{\varepsilon} = -e^{-\delta} \sum_{k=1}^\infty \phi_{\lambda, p, \delta, \rho}(n, x_k).$$

Provided $\delta > 0$, a similar argument can be used to produce the same expression for the left partial derivative. $\qquad \square$

Note that for any $q > 0$ and fixed n, using (2.1) we can find a distance r such that the probability that there exists a site at distance more than r from $R(2n, \rho)$ that affects some site within $R(2n, \rho)$ is less than q. Hence for δ sufficiently small we have

$$\mathbb{P}_{\lambda, p, 0}(H_{n, \rho}) - \mathbb{P}_{\lambda, p, \delta}(H_{n, \rho}) \leq q + (1 - e^{-\delta})(6n + 2r)^2 \leq 2q,$$

and thus $\mathbb{P}_{\lambda, p, \delta}(H_{n, \rho})$ is right continuous in δ at $\delta = 0$.

We seek to bound the effect of a slight change in δ in terms of the effect of a change in p. To do this we shall use a variant of arguments from [12]. Let y be an odd site and let $r \in \mathbb{N}$; then define $C_r = C_r(y)$ be the square of side length $2r + 1$ centred at y. Define $E_\rho(n, y, r)$ to be the event that if we use $t_y = 0$ then (i) the event $H_{n, \rho}$ occurs if we change the colour of all the sites in C_r to black (and leaving other sites unchanged) and (ii) the event $H_{n, \rho}$ does not occur if we change the colour of all the sites in C_r to white.

Lemma 4.1. *Let $\varepsilon \in (0, 1/2)$, $\rho \in \mathbb{N}$. There exists a constant $c_2 = c_2(\varepsilon, \rho) \in \mathbb{R}_+$ such that for any odd $y \in \mathbb{Z}^2$, any $n \in \mathbb{N}$, and*

$$(\lambda, p, \delta) \in [\varepsilon, 1/\varepsilon] \times [\varepsilon, 1 - \varepsilon] \times [0, 1]$$

we have

$$\phi_{\lambda, p, \delta, \rho}(n, y) \leq \mathbb{P}_{\lambda, p, \delta}[E_\rho(n, y, 1)] + \sum_{r=1}^\infty \frac{c_2^r \mathbb{P}_{\lambda, p, \delta}[E_\rho(n, y, r + 1)]}{\lfloor r/2 \rfloor!}.$$

Proof. The proof is similar to that of Lemma 5.2 of [12]. For $r \in \mathbb{N}$ let $\tilde{E}_\rho(n, y, r)$ be the event that (i) y is pivotal for event $H_{n, \rho}$ and (ii) event $H_{n, \rho}$ occurs when we use the arrival time $t_y = 0$ but then change all sites in C_r to black. As in [12], it is sufficient to prove that there is a constant c_2 such that

$$\leq \mathbb{P}_{\lambda, p, \delta}[\tilde{E}_\rho(n, y, r + 1) \setminus \tilde{E}_\rho(n, y, r)] \leq \frac{c_2^r \mathbb{P}_{\lambda, p, \delta}[E_\rho(n, y, r + 1)]}{\lfloor r/2 \rfloor!}, \quad r \in \mathbb{N}.$$

This is proved in the same manner as the corresponding equation (5.6) of [12]. In short, the idea is to define the event $F(r)$ that y affects some site outside $C(r)$; to observe that

$$\tilde{E}_\rho(n, y, r + 1) \setminus \tilde{E}_\rho(n, y, r) \subset F(r) \cap E_\rho(n, y, r + 1),$$

and then to use a coupling device to show that there is a constant c such that

$$\mathbb{P}_{\lambda, p, \delta}[E_\rho(n, y, r + 1) \cap F(r)] \leq c^r \mathbb{P}_{\lambda, p, \delta}[E_\rho(n, y, r + 1)] \mathbb{P}_{\lambda, p, \delta, \rho}[F(r)].$$

For details, see [12].

$\qquad \square$

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For $y \in \mathbb{Z}^2$, let $z_\rho(n, y)$ be the nearest even site in $R(2(n-3), \rho)$ to y using Euclidean distance, taking the first according to the lexicographic ordering when there is a choice. Let $z'_\rho(n, y) := z_\rho(n, y) + (1/2, 1/2)$.

Lemma 4.2. *Let $\varepsilon \in (0, 1/2)$, $\rho \in \mathbb{N}$. There exists a constant $c_3 = c_3(\varepsilon, \rho) \in (0, \infty)$ such that for any odd $y \in \mathbb{Z}^2$, any $n \in \mathbb{N}$ with $n \geq 60$, and any $(\lambda, p, \delta, r) \in [\varepsilon, 1/\varepsilon] \times [\varepsilon, 1 - \varepsilon] \times [0, 1] \times \mathbb{N}$, we have that*

$$\mathbb{P}_{\lambda,p,\delta}[E_\rho(n, y, r)] \leq c_3^r \phi_{\lambda,p,\delta}(n, z'_\rho(n, y)) \mathbf{1}_{R(2(n+r),\rho)}(y).$$

Proof. This is proved in the same manner as Proposition 5.1 of [12]. The only differences compared with that result is that here we consider crossing of the rectangle $R(2n, \rho)$ whereas in [12] it was the square $R(2n, 1)$, and that here y is odd whereas in [12] y is even. These have little effect on the argument.

The idea of the argument is as follows. Suppose y is such that $B(2r)$ is contained in $R(2n, \rho)$ (the other case is considered separately but the argument is not dissimilar in that case). If the event $E_\rho(n, y, r)$ occurs then there exist disjoint black paths from the left and right sides of $R(2n, \rho)$ to the boundary of $B(2r)$. One can establish existence of a collection of $O(r)$ sites in $B(2r)$, such that if we resample the arrival times and enhancement variables inside $B(2r)$ (but change nothing outside $B(2r)$), then given conditions on the resampled outcomes at this set of sites we will have $z'_\rho(n, y)$ being pivotal. For further (quite lengthy) details, see [12]. $\qquad\square$

Lemma 4.3. *For any $\varepsilon \in (0, 1/2)$, $\rho \in \mathbb{N}$, there is a constant $c_4 = c_4(\varepsilon, \rho)$ such that for any odd y, $n \in \mathbb{N}$, and $(\lambda, p, \delta) \in [\varepsilon, 1/\varepsilon] \times [\varepsilon, 1 - \varepsilon] \times [0, 1]$, we have*

$$\sum_{y \in \mathbb{Z}^2 : y \text{ odd}} \phi_{\lambda,p,\delta,\rho}(n, y) \leq c_4 \sum_{z \in R(2(n-3),\rho) : z \text{ even}} \phi_{\lambda,p,\delta,\rho}(n, z').$$

Proof. Using our Lemmas 4.1 and 4.2, the proof is as in the first step of the proof of Proposition 3.1 in [12]. In this case, this step is just a few lines; one groups the terms in the sum on the left according to those y for which $z_\rho(y)$ takes the value of a particular term z in the sum on the right. $\qquad\square$

Corollary 4.1. *For any $\varepsilon \in (0, 1/2)$, $\rho \in \mathbb{N}$ there exists a constant $c_5 = c_5(\varepsilon, \rho)$ such that for any $n \in \mathbb{N}$, and $(\lambda, p, \delta) \in [\varepsilon, 1/\varepsilon] \times [\varepsilon, 1 - \varepsilon] \times [0, 1]$, we have*

$$\left| \frac{\partial h_\rho(n, \lambda, p, \delta)}{\partial \delta} \right| \leq c_5 \frac{\partial h_\rho(n, \lambda, p, \delta)}{\partial p}. \tag{4.6}$$

Proof. The result follows immediately from Lemma 4.3 and Proposition 4.1. $\qquad\square$

5 Proof of Proposition 2.1

Proposition 2.1 says that the effect on the crossing probability $h_\rho(n, \lambda, p, \delta)$ of a small change in λ, is comparable to the effect of a small change in p. To prove this, we need to find an appropriate inequality connecting even sites being pivotal, and diamond sites being pivotal. Figure 2 demonstrates one of the four possible arrangements of occupied

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sites closest to the diamond site in question (the other possibilities being the reflection of the occupation locations and colour inversions of these two). In order for this diamond site to be pivotal, in addition to the sites locally having an arrangement of this form we also require that there be a black path from the left edge of the rectangle to one of the occupied black sites close to the diamond site, a black path from the right edge to the other occupied black site, a white path from the top edge to one of the occupied white sites, and a white path from the bottom edge to the other occupied white site.
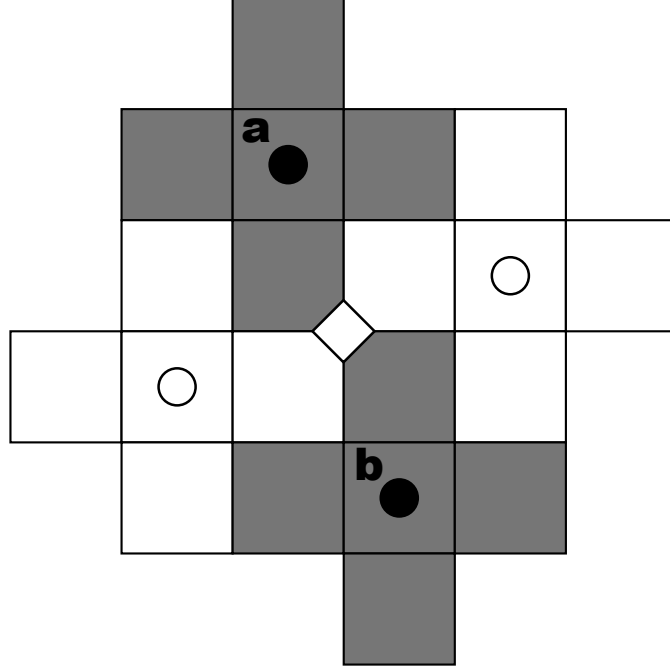

Figure 2: An example of the possible local arrangements of occupied and blocked sites such that a diamond site may be pivotal.

Recall the definition that for any octagon site y, the site y' is the site $y + (1/2, 1/2)$, and define similarly y'' as the site $y + (1/2, -1/2)$.

Lemma 5.1. *For any $\varepsilon \in (0,1)$ there is a constant $c_6 = c_6(\varepsilon) > 0$, such that for any $\lambda \in [\varepsilon, 1/\varepsilon]$, and $p \in [0,1]$, $n \in \mathbb{N}$, and any even $y \in \mathbb{Z}^2$ we have*

$$\phi_{\lambda,p,0,3}(n, y') \leq c_6 \phi_{\lambda,p,0,3}(n, y), \tag{5.1}$$

and

$$\phi_{\lambda,p,0,3}(n, y'') \leq c_6 \phi_{\lambda,p,0,3}(n, y). \tag{5.2}$$

Proof. Fix an even site y, and let $\mathcal{S}_y = (S_x)_{x \in \Lambda}$ be the collection of arrival times and enhancement variables in one eRSA process. In a similar manner to the proof of Proposition 5.1 of [12], we shall construct a coupled process $\mathcal{U}_y = (U_x)_{x \in \Lambda}$. For $n \in \mathbb{N}$, let $B(2n+1)$ be the collection of octagon and diamond sites within $[-n, n] \times [-n, n]$. Let $\mathcal{S}_y$ be as above, let $\mathcal{T}_y = (T_x)_{x \in \Lambda}$ be the set of arrival times and enhancement variables in an independent RSA process, and let $\mathcal{B} = (B_x)_{x \in \Lambda}$ be a collection of independent Bernoulli

random variables with parameter 0.5. Then we define

$$
\begin{aligned}
U_x \quad &= S_x, && x \in \mathbb{Z}^2 \setminus (B(13) + y); \\
&= T_x, && x \in \mathbb{Z}^2 \cap (B(7) + y); \\
&= B_x T_x + (1 - B_x) S_x, && x \in \mathbb{Z}^2 \cap ((B(13) \setminus B(7)) + y); \\
&= S_x, && x - (1/2, 1/2) \in \mathbb{Z}^2.
\end{aligned}
$$

We also define an independent exponential random variable T with parameter λ.

We now define three events denoted E_1, E_2 and E_3, such that if all three events hold, then the site y is pivotal in the $\mathcal{U}_y$ process. Let E_1 be the event that the diamond site y' is pivotal for the $\mathcal{S}_y$ process. For $m < n$, let $A_y(m, n)$ be the square annulus $y + B(n) \setminus B(m)$. We shall define E_2 to be an event concerning sites in the annulus $A_y(3, 13)$ which ensures that for the $\mathcal{U}_y$ process the occupied octagon sites of the $\mathcal{S}_y$ process therein have an earlier arrival time than all of their neighbours in that annulus, and moreover all occupied sites in $A_y(3, 7)$ have arrival times between 0.5 and 1. Define E_2 as follows:

$$
\begin{aligned}
E_2 = \bigcap \; &\cap_{\{x \in A_y(7,13) \cap \mathbb{Z}^2 : S_x > 1 \text{ and } x \text{ is occupied in } \mathcal{S}_y\}} \{B_x = 1 \text{ and } T_x < 1\} \\
&\cap_{\{x \in A_y(7,13) \cap \mathbb{Z}^2 : S_x < 1 \text{ and } x \text{ is blocked in } \mathcal{S}_y\}} \{B_x = 1 \text{ and } T_x > 1\} \\
&\cap_{\{x \in A_y(7,13) \cap \mathbb{Z}^2 : S_x < 1 \text{ and } x \text{ is occupied in } \mathcal{S}_y\}} \{B_x = 0\} \\
&\cap_{\{x \in A_y(7,13) \cap \mathbb{Z}^2 : S_x > 1 \text{ and } x \text{ is blocked in } \mathcal{S}_y\}} \{B_x = 0\} \\
&\cap_{\{x \in A_y(3,7) \cap \mathbb{Z}^2 : x \text{ is occupied in } \mathcal{S}_y\}} \{0.5 < T_x < 1\} \\
&\cap_{\{x \in A_y(3,7) \cap \mathbb{Z}^2 : x \text{ is blocked in } \mathcal{S}_y\}} \{T_x > 1\}.
\end{aligned}
$$

We shall define E_3 to be an event concerning the sites in $y + B(3)$ which ensures (in conjunction with E_2) that the sites next to y will become occupied if the arrival at y is delayed but blocked if the arrival at y is not delayed. To be precise, define

$$
\begin{aligned}
E_3 := \{T_y \leq 0.1\} \cap_{z \in y + B(3) : z \text{ odd}} \{0.1 < T_z \leq 0.2\} \cap \\
\cap_{z \in y + B(3) : z \text{ even}, z \neq y} \{0.2 < T_z \leq 0.3\} \cap \{T > 0.2\}.
\end{aligned}
$$

Consider the state of the $\mathcal{U}_y$ process if all of these events occur. If E_2 and E_3 both hold, then every even octagon site within the square $y + B(3)$ is occupied if we have the arrival time at y being T_y, but blocked if we delay the arrival at y by T. As noted in Lemma 5.1 of [12], provided E_2 occurs then the states of sites outside $y + B(7)$ in the $\mathcal{U}_y$ process match the states of those sites in the $\mathcal{S}_y$ process. Now we consider any even octagon site within $A_y(3, 7)$. If this site was black in the $\mathcal{S}_y$ process, then in the $\mathcal{U}_y$ process it has arrival time less than 1 and any adjacent sites outside $y + B(3)$ have arrival times at least 1, thus are unable to block it. Since all odd sites within $y + B(3)$ are blocked by the arrival at y, it follows that the site under consideration has first arrival time strictly lower than all adjacent unblocked sites and hence is occupied.

Suppose y' is pivotal in $\mathcal{S}_y$. Without loss of generality, we assume that in the $\mathcal{S}_y$ process the local arrangement of occupied sites at y' matches that in figure 2 and that the site labelled a has a black path connecting it to the left side of the rectangle, and that the site labelled b has a black path connecting it to the right side of the rectangle. By our argument and due to black paths being increasing in black sites, it follows that in

the U_y process there is a black path from the left side of the rectangle to a, from the site a to the site b due to all the sites in the square $y + B(3)$ being black, and from the site b to the right side of the rectangle. As such, we see that in the $\mathcal{U}_y$ process, if the events E_1, E_2 and E_3 hold and we take T_y as the arrival time at y we have a horizontal black crossing of the rectangle.

A similar argument shows that if we delay the arrival at y by the random variable T and the events E_1, E_2 and E_3 hold then we have a vertical white crossing of the rectangle, and thus the site y is pivotal. We then obtain (5.1) by noting that the events $E_1 \cap E_2$ and E_3 are independent, that the probability of E_1 is $\phi_{\lambda,p,0,3}(n, y')$, and that there is a strictly positive lower bound both on $\mathbb{P}_{\lambda,p,0}[E_2|E_1]$ and on $\mathbb{P}_{\lambda,p,0}[E_3]$, uniformly over $0 \leq p \leq 1$ and $\varepsilon \leq \lambda \leq 1/\varepsilon$, and over outcomes of the $\mathcal{S}_y$ process in event E_1.

A similar argument provides the second inequality (5.2). $\qquad\square$

Proof of Proposition 2.1. The second inequality of (2.2) follows immediately from Lemma 5.1, equation (4.2) and equation (4.1).

The first inequality of (2.2) is obtained as in the proof of Proposition 3.1 of [12]. $\quad\square$

Remark. The proof of Lemma 5.1 (and hence, of the second inequality of (2.2)) is simpler than the the proof of Proposition 3.1 of [12], required for the proof of the first inequality of (2.2). This is because in proving Lemma 5.1 we change a configuration with a pivotal diamond site so that a neighbouring octagon site is pivotal, and can arrange that changing the arrival time at the octagon site affects the nearby sites in a manner which helps to make it pivotal. For the inequality the other way, we need to change a configuration with a pivotal octagon site to make a neighbouring diamond site pivotal, which is more complicated since the diamond site has no effect on other sites, so we need to change the configuration of states of nearby octagon sites 'by hand' to make the diamond site pivotal.

6 Proof of Proposition 2.2

To prove Proposition 2.2 we shall use Proposition 3.1, our sharp thresholds result. Since that result refers to a discrete product space, we shall need to discretise time, and also transfer the model to a torus to achieve the symmetry needed for applying Proposition 3.1.

Given $n \in \mathbb{N}$, let $\mathbb{T}(2n)$ denote the torus formed from a $2n$ by $2n$ square of octagon sites and the diamond sites at the upper right corner of each octagon site. We shall arbitrarily choose an octagon site in the torus to be the origin, and from this we can define even and odd sites on $\mathbb{T}(2n)$ and hence have enhanced RSA as before on the torus. Where required, we shall denote by $P_{\lambda,p}^{\mathbb{T}(2n)}$ and $P_{\lambda,p}^{\Lambda}$ the probability measures for enhanced RSA with parameters λ and p on the torus $\mathbb{T}(2n)$ and on the full enhanced integer lattice Λ respectively.

Lemma 6.1. *Let $n \in \mathbb{N}$, $\lambda > 0$ and $p \in (0,1)$, and let R be a rectangle with long side length at most $2n - 4\sqrt{2n}$. Then*

$$\left| P_{\lambda,p}^{\mathbb{T}(2n)}[H(R)] - P_{\lambda,p}^{\Lambda}[H(R)] \right| < e(n)$$

where $H(R)$ is the event that R has a horizontal black crossing and $e(n)$ is some $o(1)$ function independent of R.

Proof. We can couple enhanced RSA on Λ and on $\mathbb{T}(2n)$ such that the arrival times at integer sites and colours of diamond sites agree on $\{(a,b) : 0 \leq a, b \leq 2n\}$. By Lemmas 2.1 and 2.4 the probability that there is a site within a rectangle contained within $\{(a,b) : 2\lceil\sqrt{2n}\rceil \leq a, b \leq 2n - 2\lceil\sqrt{2n}\rceil\}$ whose colour disagrees with the colour of the associated site in Λ tends to 0 as $n \to \infty$, and so the result follows. $\square$

Given $n \in \mathbb{N}$, define the torus $\mathbb{T} = \mathbb{T}(20n)$. Let Q_n be the $20n$ by $20n$ square region in Λ which is identified with $\mathbb{T}$. Also let Q_n^e be the set of even sites in Q_n and let $\mathbb{T}^e$ be the set of even sites in $\mathbb{T}$. Set $\delta := \delta(n) := (\log n)^{-1/2}$.

Given $(\lambda_0, \tilde{p}, \lambda_1) \in \mathbb{R}_+ \times (0,1) \times \mathbb{R}_+$, let $\mathbb{P}^{\mathbb{T}}_{\lambda_0, \tilde{p}, \lambda_1}$ be the probability measure associated with the enhanced RSA model on the torus $\mathbb{T}$ with arrivals rate λ_0 at even sites and λ_1 at odd sites, and diamond sites black with probability $\tilde{p}$. (When $\lambda_1 = 1$, we sometimes omit the third subscript λ_1 from the notation.) We now construct a discrete-time version of this process. At each site $x \in \mathbb{Z}^2$ we shall divide the time-axis into blocks of length δ, and discarding all blocks that had their start time later than n we have a product space $\tilde{\mathbb{T}} := \mathbb{T} \times \{-1, 0, 1, 2, \ldots, \lfloor n/\delta \rfloor\}$ where $(x, -1)$ represents the diamond site $x' := x + (1/2, 1/2)$, and (x, k) for $k \in \{0, 1, \ldots, \lfloor n/\delta \rfloor\}$ represents the site x at times in the interval $I_\delta(k) := [k\delta, (k+1)\delta)$. We denote the probability measure on this new space by $\mathbb{P}^{\tilde{\mathbb{T}}}_{\lambda_0, \tilde{p}, \lambda_1}$.

We shall now construct a random field $\mathbf{X} = (X(x,k) : (x,k) \in \tilde{\mathbb{T}})$ with each $X(x,k)$ taking values in $\{0, 1, 2, 3\}$. For an even site x and for $k \geq 0$, we set $X(x,k) = 3$ if there is an attempted arrival at x within $I_\delta(k)$ and $X(x,k) \in \{0, 1, 2\}$ if not. For an odd site x and for $k \geq 0$, we set $X(x,k) = 0$ if there is an attempted arrival at x within $I_\delta(k)$ and $X(x,k) \in \{1, 2, 3\}$ if not. For any site x, we put $X(x, -1) \in \{2, 3\}$ if x' is black, and $X(x, -1) \in \{0, 1\}$ if x' is white. To construct a representation of this model in a discrete product space we consider all arrivals at a site instead of solely the first, so that $X(x, k_1)$ is independent of $X(x, k_2)$ whenever $k_1 \neq k_2$. Where there is a choice of the value of $X(z)$ for $z \in \tilde{\mathbb{T}}$, we choose randomly and independently of $X(z')$ for all $z' \neq z$ so that the distribution of $\mathbf{X}$, denoted $\mathbb{P}^{\tilde{\mathbb{T}}}_{\lambda_0, \tilde{p}, \lambda_1}$, satisfies

$$\mathbb{P}^{\tilde{\mathbb{T}}}_{\lambda_0, \tilde{p}, \lambda_1}(X(z) = 3) = 1 - e^{-\lambda_0 \delta}; \qquad (6.1)$$
$$\mathbb{P}^{\tilde{\mathbb{T}}}_{\lambda_0, \tilde{p}, \lambda_1}(X(z) = 2) = \tilde{p} + e^{-\lambda_0 \delta} - 1;$$
$$\mathbb{P}^{\tilde{\mathbb{T}}}_{\lambda_0, \tilde{p}, \lambda_1}(X(z) = 1) = e^{-\lambda_1 \delta} - \tilde{p};$$
$$\mathbb{P}^{\tilde{\mathbb{T}}}_{\lambda_0, \tilde{p}, \lambda_1}(X(z) = 0) = 1 - e^{-\lambda_1 \delta}.$$

Since we assume $\tilde{p} \in (0,1)$, for large enough n these really are probabilities.

Let $E^n_{\text{fast}}(\mathbb{T}^e)$ be the event that for all $z \in \mathbb{T}^e$ the first arrival time at z is less than $\sqrt{n}$, and let $E^n_{\text{fast}}(Q_n^e)$ be defined similarly.

Let E_n be the event that there is some $18n$ by $2n$ rectangle in $\mathbb{T}$ with a horizontal black crossing after the arrival times at all even sites are delayed by 2δ and that also $E^n_{\text{fast}}(\mathbb{T}^e)$ occurs. Let E_n^{crude} be the event that the state of $\mathbf{X} := \{X(x,k) : (x,k) \in \tilde{\mathbb{T}}\}$ is such that E_n is possible given $\mathbf{X}$; this can be seen as either an event on the discrete

time torus $\tilde{\mathbb{T}}$, or as an event on the continuous time torus representing that the state of $\mathbf{X}$ consistent with the arrival times satisfies the understanding of E_n^{crude} above.

Lemma 6.2. *Let $\lambda > 0$, $p \in (0,1)$ and $\varepsilon \in (0, 1-p)$. Suppose for some $\rho > 0$ that $\limsup_{n \to \infty} h_\rho(n, \lambda, p) > 0$. Then*

$$\limsup_{n \to \infty} \mathbb{P}^{\tilde{\mathbb{T}}}_{\lambda, p+\varepsilon/2, 1}(E_n^{\text{crude}}) > 0. \tag{6.2}$$

Proof. Let $R_n := [1, 18n] \times [1, 2n]$, considered as a rectangle in the torus $\mathbb{T}$. Let D_n be the event that R_n has a horizontal black crossing after the arrival times at all even sites are delayed by 2δ. By the union bound and the exponential decay of the tail of the exponential distribution, we have $\lim_{n \to \infty} \mathbb{P}^{\Lambda}_{\lambda, p+\varepsilon/2}(E_{\text{fast}}^n(Q_n^e)) = 1$. Hence using Lemma 2.4, letting E_{dense}^n be the event $E_{\text{dense}}(R_n, 2\lceil \sqrt{2n} \rceil)$ we have

$$\mathbb{P}^{\mathbb{T}}_{\lambda, p+\varepsilon/2}[D_n \cap E_{\text{fast}}^n(\mathbb{T}^e)] \geq \mathbb{P}^{\Lambda}_{\lambda, p+\varepsilon/2}[D_n \cap E_{\text{dense}}^n \cap E_{\text{fast}}^n(Q_n^e)]$$
$$= \mathbb{P}^{\Lambda}_{\lambda, p+\varepsilon/2}(D_n) + o(1).$$

Then using Corollary 4.1 and the Mean Value Theorem, followed by Lemma 2.6, we obtain

$$\limsup_{n \to \infty} \mathbb{P}^{\mathbb{T}}_{\lambda, p+\varepsilon/2}[D_n \cap E_{\text{fast}}^n(\mathbb{T}^e)] \geq \limsup_{n \to \infty} \mathbb{P}^{\Lambda}_{\lambda, p}(H_{n,9})$$
$$> 0. \tag{6.3}$$

Clearly $\mathbb{P}^{\mathbb{T}}_{\lambda, p+\varepsilon/2}(E_n) \geq \mathbb{P}^{\mathbb{T}}_{\lambda, p+\varepsilon/2}[D_n \cap E_{\text{fast}}^n(\mathbb{T}^e)]$. Since for any $(\lambda_0, \tilde{p}, \lambda_1)$ we have

$$\mathbb{P}^{\tilde{\mathbb{T}}}_{\lambda_0, \tilde{p}, \lambda_1}(E_n^{\text{crude}}) \geq \mathbb{E}\left[\mathbb{P}^{\mathbb{T}}_{\lambda_0, \tilde{p}, \lambda_1}(E_n | \mathbf{X})\right] = \mathbb{P}^{\mathbb{T}}_{\lambda_0, \tilde{p}, \lambda_1}(E_n),$$

from (6.3) we have (6.2). $\qquad \square$

We now we use our sharp thresholds result to show that after a slight adjustment of parameters, the probability of the discrete event E_n^{crude} is infinitely often close to 1 rather than just being bounded away from zero as in (6.2).

Lemma 6.3. *Under the assumptions of Lemma 6.2,*

$$\limsup_{n \to \infty} \mathbb{P}^{\tilde{\mathbb{T}}}_{(1+\varepsilon)^{1/2}\lambda, p+\varepsilon/2, (1+\varepsilon)^{-1/2}}(E_n^{\text{crude}}) = 1. \tag{6.4}$$

Proof. Set $N := N(n) := 400n^2(2 + \lfloor \frac{n}{\delta} \rfloor)$. Given a probability vector $\boldsymbol{p'} = (p_0', p_1', p_2', p_3')$, we define the probability measure $\mathbb{P}^n_{\boldsymbol{p'}}$ on the space $\{0,1,2,3\}^N$ as in Section 3. We can now think of E_n^{crude} as being an event E_n^{disc} in $\{0,1,2,3\}^N$, by enumerating the (x,k) pairs as $z_1, z_2, \ldots, z_N$ and identifying the value of $\mathbf{X}$ with an element of $\{0,1,2,3\}^N$. Given $\lambda_0, \tilde{p}$ and λ_1, the distribution of $\mathbf{X}$ under this identification is given by $\mathbb{P}^N_{\boldsymbol{p}}$ with the entries of $\boldsymbol{p}$ given by (6.1).

The event E_n is symmetric under the group of permutations of sites by translations of the torus (modulo $20n$) that send even sites to even sites, and therefore so too are E_n^{crude} and E_n^{disc}. This group of permutations has order $200n^2$.

We claim that E_n^{disc} is increasing in $\mathbf{X}$. Indeed, suppose $z = (x,k) \in \tilde{\mathbb{T}}$. If $k = -1$ then z corresponds to the diamond site x', and an increase in $X(z)$ corresponds either to

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leaving x' unchanged, or to changing x' from being white to being black. If $k \geq 0$ and x is an odd site, an increase in $X(z)$ from 0 corresponds to removing any arrivals at x in the time period $I_\delta(k)$ and otherwise leaving things unchanged. If $k \geq 0$ and x is an even site, an increase in $X(z)$ corresponds to either leaving things unchanged, or adding an arrival at x in the time period $I_\delta(k)$. Thus regardless of the nature of a site z, E_n^{disc} is increasing in $X(z)$.

In order to apply Proposition 3.1, we compare two models, i.e. two probability vectors (p_0, p_1, p_2, p_3) and (q_0, q_1, q_2, q_3), where p_i is the probability that $X(z) = i$ in the first model, and q_i is the probability that $X(z) = i$ in the second model. Our first model has parameters $\lambda_0 = \lambda$, $\lambda_1 = 1$, and $\tilde{p} = p + \varepsilon/2$, while our second model has parameters $\lambda_0 = (1+\varepsilon)^{1/2}\lambda$, $\lambda_1 = (1+\varepsilon)^{-1/2}$ and $\tilde{p} = p + \varepsilon$. Then using (6.1) we have

$$
\begin{aligned}
p_3 &= 1 - e^{-\lambda\delta}, & q_3 &= 1 - e^{-(1+\varepsilon)^{1/2}\lambda\delta}; \\
p_2 &= e^{-\lambda\delta} + \varepsilon/2 + p - 1, & q_2 &= e^{-(1+\varepsilon)^{1/2}\lambda\delta} + \varepsilon + p - 1; \\
p_1 &= e^{-\delta} - \varepsilon/2 - p, & q_1 &= e^{-(1+\varepsilon)^{-1/2}\delta} - \varepsilon - p; \\
p_0 &= 1 - e^{-\delta}, & q_0 &= 1 - e^{-(1+\varepsilon)^{-1/2}\delta}.
\end{aligned}
$$

From the equivalence of E_n^{disc} and E_n^{crude}, and Lemma 6.2, we have

$$
\limsup_{n\to\infty} \mathbb{P}^N_{p_0,p_1,p_2,p_3}(E_n^{\mathrm{disc}}) = \limsup_{n\to\infty} \mathbb{P}^{\tilde{\mathbb{T}}}_{\lambda,p+\varepsilon/2,1}(E_n^{\mathrm{crude}}) > 0.
$$

We shall now apply Proposition 3.1. Note that

$$
\begin{aligned}
q_3 - p_3 &= e^{-\lambda\delta} - e^{-(1+\varepsilon)^{1/2}\lambda\delta} \sim \delta((1+\varepsilon)^{1/2} - 1)\lambda; \\
p_1 - q_1 &= \varepsilon/2 + e^{-\delta} - e^{-(1+\varepsilon)^{-1/2}\delta} \to \varepsilon/2; \\
p_0 - q_0 &= e^{-(1+\varepsilon)^{-1/2}\delta} - e^{-\delta} \sim \delta(1 - (1+\varepsilon)^{-1/2}) \\
&= \delta\frac{(1+\varepsilon)^{1/2} - 1}{(1+\varepsilon)^{1/2}}.
\end{aligned}
$$

Set $\gamma = \min(p_0 - q_0, q_3 - p_3)$. For sufficiently high n, we obtain that $p_0 > q_0$, $p_1 > q_1$ and $q_3 > p_3$. Hence $\gamma > 0$ and (q_0, q_1, q_2, q_3) dominates $(p_0 - \gamma, p_1, p_2, p_3 + \gamma)$. We shall apply Proposition 3.1 with $k = 3$. In the terminology of that result, we have $q_{\max} = \min(p_2, p_1)$. Fix

$$
\eta \in (0, \limsup_{n\to\infty} \mathbb{P}^N_{p_0,p_1,p_2,p_3}(E_n^{\mathrm{disc}})).
$$

Since $p\log(4/p)$ takes maximum value $\log(4) < 2$, the right hand side of (3.1) is at most $4000\log(1/\eta)$. Since $\delta = (\log n)^{-1/2}$, for n large enough we have $\gamma\log(200n^2) > 4000\log(1/\eta)$. Thus Proposition 3.1 is applicable; by that result, and the equivalence of E_n^{disc} and E_n^{crude}, for infinitely many n we have

$$
\mathbb{P}^{\tilde{\mathbb{T}}}_{(1+\varepsilon)^{1/2}\lambda,p+\varepsilon,(1+\varepsilon)^{-1/2}}(E_n^{\mathrm{crude}}) = \mathbb{P}^N_{q_0,q_1,q_2,q_3}(E_n^{\mathrm{disc}}) > 1 - \eta,
$$

and (6.4) follows. $\qquad\square$

Lemma 6.4. *Let $\lambda > 0$, $p \in (0, 1)$ and $\varepsilon \in (0, 1 - p)$. Suppose for some $\rho > 0$ that $\limsup_{n \to \infty} h_\rho(n, \lambda, p) > 0$. Then*

$$\limsup_{n \to \infty} h_3(n, \lambda(1 + \varepsilon), p + \varepsilon) = 1. \tag{6.5}$$

Recall that our goal is to prove Proposition 2.2, which gives a similar conclusion for $h_3(n, \lambda + \varepsilon, p)$ and $h_3(n, \lambda, p + \varepsilon)$. Thus with this lemma, we are nearly there.

Proof of Lemma 6.4. Let F_n be the event that there is a horizontal black crossing of some $18n$ by $2n$ rectangle in $\mathbb{T}$ (like E_n but with no time delay and with no requirement for the event $E_{\mathrm{fast}}^n(\mathbb{T}^e)$ to occur). We assert the event inclusion $E_n^{\mathrm{crude}} \subset F_n$.

Indeed, consider any state $\mathbf{X}_0 \in E_n^{\mathrm{crude}}$. Let $x_0, x_1, \ldots$ be an enumeration of the sites of $\mathbb{T} \cap \mathbb{Z}^2$, let $\mathrm{col}_{x'}$ denote the colour of the diamond site x', and let $Z_1 = (\mathrm{col}_{x'_0}, t_{x_0}, \mathrm{col}_{x'_1}, t_{x_1}, \ldots)$ be a collection of arrival times at octagon sites and colours of diamond sites on the torus which induces state $\mathbf{X}_0$ and such that E_n holds. By definition, such a Z_1 exists. Let Z_2 be any other collection of octagon site arrival times and diamond site colours with state consistent with $\mathbf{X}_0$. At each even site of the torus, the first arrival time under Z_2 can be at most δ later than the first arrival at that site in Z_1, and similarly the first arrival at an odd site in Z_2 can be no more than δ earlier than the first arrival in Z_1. Therefore any sites which are black when all the arrival times at even sites in Z_1 are delayed by 2δ (as per the definition of E_n) are also black in Z_2 (with no delay). Since the existence of a horizontal crossing is black-increasing, and since Z_1 with a 2δ delay on the arrival time at even sites has a horizontal black crossing of some $18n$ by $2n$ rectangle, Z_2 must therefore have a horizontal black crossing of the same $18n$ by $2n$ rectangle. Hence F_n occurs and our assertion is justified.

Suppose for some $\rho > 0$ that $\limsup_{n \to \infty} h_\rho(n, \lambda, p) > 0$. Let $\varepsilon_1 > 0$. By time rescaling $\mathbb{P}^{\mathbb{T}}_{\lambda(1+\varepsilon), p+\varepsilon, 1}(F_n) = \mathbb{P}^{\mathbb{T}}_{\lambda(1+\varepsilon)^{1/2}, p+\varepsilon, (1+\varepsilon)^{-1/2}}(F_n)$. Hence by the event inclusion just proved, and Lemma 6.3, we have infinitely often (i.e., for infinitely many n) that

$$\mathbb{P}^{\mathbb{T}}_{\lambda(1+\varepsilon), p+\varepsilon, 1}(F_n) = \mathbb{P}^{\mathbb{T}}_{\lambda(1+\varepsilon)^{1/2}, p+\varepsilon, (1+\varepsilon)^{-1/2}}(F_n) > 1 - \varepsilon_1. \tag{6.6}$$

Now cover $\mathbb{T}$ with a set of $12n$ by $4n$ rectangles $R_{n,1}, \ldots, R_{n,40}$ such that whenever F_n holds there is a black path crossing some $R_{n,i}$ horizontally. We can do this by using rectangles with lower left corner having x-coordinate a multiple of $5n$ and y-coordinate a multiple of $2n$. Let $H_{n,i}$ be the event that $R_{n,i}$ has a black horizontal crossing, and note that $H_{n,i}^c$ is white-increasing. Using the Harris-FKG inequality (Lemma 2.2), followed by Lemma 6.1, we have

$$\mathbb{P}^{\mathbb{T}}_{\lambda(1+\varepsilon), p+\varepsilon, 1}\left(\bigcap_{i=1}^{40} H_{n,i}^c\right) \geq \prod_{i=1}^{40} \mathbb{P}^{\mathbb{T}}_{\lambda(1+\varepsilon), p+\varepsilon, 1}(H_{n,i}^c)$$
$$= (\mathbb{P}^{\mathbb{T}}_{\lambda(1+\varepsilon), p+\varepsilon, 1}(H_{n,i}^c))^{40}$$
$$= (1 - h_3(2n, \lambda(1 + \varepsilon), p + \varepsilon) + o(1))^{40}.$$

If none of the $H_{n,i}$ hold then F_n fails, so by (6.6), infinitely often

$$1 - h_3(2n, \lambda(1 + \varepsilon), p + \varepsilon) \leq \varepsilon_1^{1/40} + o(1),$$

and hence we have (6.5). $\qquad\square$

Proof of Proposition 2.2. Let $\lambda > 0$, $p \in (0,1)$ and $\varepsilon \in (0, 1-p)$. Suppose for some $\rho > 0$ that $\limsup_{n\to\infty} h_\rho(n, \lambda, p) > 0$. Choose $\varepsilon_2 > 0$ such that $\varepsilon_2 < \lambda$ and $\lambda + 1 < \varepsilon_2^{-1}$, and $\varepsilon_2 < p$ and $p + \varepsilon < 1 - \varepsilon_2$. Let $c_1 = c_1(\varepsilon_2)$ be as in Proposition 2.1 and assume without loss of generality that $c_1 \geq 1$. Then by the first inequality of (2.2), for all n we have

$$h_3(n, \lambda, p + \varepsilon) \geq h_3(n, \lambda + \varepsilon/(2c_1), p + \varepsilon/2);$$

hence by Lemma 6.4, we have (2.4). We prove (2.3) similarly, now using the second inequality of (2.2). $\qquad\square$

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A   Proof of Lemma 3.2

By a continuity argument, it suffices to prove the result for the case where all entries of $\boldsymbol{p}$ are dyadic rationals, i.e. to show that for any n, f, a and any $\boldsymbol{p}$ with all entries dyadic rationals satisfying (3.5) we have (3.6).

Choose such a $\boldsymbol{p}$ and choose $m \in \mathbb{N}$ such that all entries of $2^m \boldsymbol{p}$ are integers. Let $\mathbb{Y}$ be the space $\{0,1\}^m$ with the uniform distribution. We identify the space $\mathbb{X} := \{0, 1, \ldots, k\}$ under measure $\mathbb{P}_{\boldsymbol{p}}$, with the space $\mathbb{Y}$, as follows. Define a function $\tau : \mathbb{Y} \to \mathbb{X}$ as follows: the first $2^m p_0$ elements of $\mathbb{Y}$ (under the upwards lexicographic ordering) are mapped to $0 \in \mathbb{X}$, the next $2^m p_1$ elements of $\mathbb{Y}$ are mapped to $1 \in \mathbb{X}$, and so on.

Using this identification, any function $g : \mathbb{X} \to \{0,1\}$ induces another function $\tilde{g} : \mathbb{Y} \to \{0,1\}$, given by $\tilde{g} = g \circ \tau$. Moreover, for $\ell \in [m]$ the influence of the ℓth coordinate of a uniform random element of $\mathbb{Y}$ on $\tilde{g}$ is equal to $w_{\ell,\boldsymbol{p}}(g)$, since switching the ℓth digit of the binary expansion of U amounts to switching the ℓth component of the corresponding random element of $\mathbb{Y}$. Writing $w(\tilde{g})$ for the sum (over ℓ) of these influences, we have by Lemma 3.1 that

$$w(\tilde{g}) \le 3k^2 p_{\max}(\boldsymbol{p}) \log(4/p_{\max}(\boldsymbol{p})) \le 3k^2 q \log(4/q). \tag{A.1}$$

We identify $\mathbb{Y}$ with the power set of $[m]$ in the natural way. For $S \in \mathbb{Y}$ (i.e. for $S \subset [m]$), we set
$$u_S(A) = (-1)^{|S \cap A|}, \quad A \subset [m].$$

It is well known (and not hard to prove) that the functions $u_S, S \subset [m]$ form an orthonormal basis of the 2^m-dimensional vector space of functions from $\mathbb{Y}$ to $\mathbb{R}$, endowed with the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle g, h \rangle = 2^{-m} \sum_{A \subset [m]} g(A) h(A).$$

Given functions h and g from $\mathbb{Y}$ to $\mathbb{R}$, define the convolution $h * g$ by

$$h * g(S) = 2^{-m} \sum_{A \subset [m]} h(A) g(S \triangle A), \quad S \subset [m], \tag{A.2}$$

where $\triangle$ denotes the symmetric difference. Also define the Walsh-Fourier transform $\hat{h}$ of h by

$$\hat{h}(S) = \langle h, u_S \rangle, \qquad S \subset [m]. \tag{A.3}$$

Associated with this is the Walsh-Fourier expansion of h, namely $h = \sum_S \hat{h}(S)u_S$, and the Parseval equation $\|h\|_2^2 := \langle h, h \rangle = \sum_S \hat{h}(S)^2$. These are both immediate from the fact that the u_S form an orthonormal basis. It is well known (and not hard to prove) that for $S \subset [m]$ we have

$$\widehat{h * g}(S) = \hat{h}(S)\hat{g}(S). \tag{A.4}$$

Define $T : \mathbb{Y} \to \mathbb{R}$ by $T(Z) = \sum_S u_S(Z)|S|^{1/2}$, for $Z \subset [m]$, where the sum is over all $S \subset [m]$. Then $\hat{T}(S) = |S|^{1/2}$ for all S. Hence by (A.4), for any $h : \mathbb{Y} \to \mathbb{R}$ we have $\widehat{T * h}(S) = \hat{h}(S)|S|^{1/2}$. Hence by the Parseval identity,

$$\|T * h\|_2^2 = \sum_{S \subset [m]} \hat{h}(S)^2 |S| = (1/4)w(h), \tag{A.5}$$

where $w(h)$ is as in (A.1) and for the last equality we have used the first paragraph of [9, p.73].

Fix $n \in \mathbb{N}$ and let $f : \mathbb{X}^n \to \{0,1\}$ be a function. Let $i \in [n]$. For $S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n \in \mathbb{Y}$, define the function $h = h[S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n] : \mathbb{Y} \to \{0,1\}$ by

$$h[S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n](S) = \tilde{f}(S_1, \ldots, S_{i-1}, S, S_{i+1}, \ldots, S_n), \tag{A.6}$$

where we set $\tilde{f}(S_1, \ldots, S_n) := f(\tau(S_1), \ldots, \tau(S_n))$. Also, define the function $v = v[S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n] : \mathbb{Y} \to \mathbb{R}$ by

$$v[S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n] = T * h[S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n].$$

Now define $W_i(S_1, \ldots, S_n) := v[S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n](S_i)$, for $S_1, \ldots, S_n \subset [m]$ (recall that we are identifying $\mathbb{Y}$ with the power set of $[m]$). Then

$$
\begin{aligned}
W_i(S_1, \ldots, S_n) &= 2^{-m} \sum_{R \subset [m]} T(R)\tilde{f}(S_1, \ldots, S_{i-1}, S_i \triangle R, S_{i+1}, \ldots, S_n) \\
&= 2^{-m} \sum_{\tilde{R} \subset [mn]} T_i(\tilde{R})\tilde{f}((S_1, \ldots, S_n) \triangle \tilde{R})
\end{aligned}
$$

where for $R_1, \ldots, R_n \subset [m]$ we set $T_i(R_1, \ldots, R_n) = T(R_i)$ if $R_j = \emptyset$ for all $j \neq i$ and $T_i(R_1, \ldots, R_n) = 0$ otherwise. Thus, with convolutions of functions on $\mathbb{Y}^n$ (or equivalently, on the power set of $[nm]$) defined analogously to (A.2), we have

$$W_i = 2^{m(n-1)}T_i * \tilde{f}. \tag{A.7}$$

For F a real-valued function on $\mathbb{Y}^n$ (or equivalently, on the power set of $[mn]$), we define the Walsh-Fourier transform of F analogously to (A.3), by $\hat{F}(S) = 2^{-nm} \sum_{B \subset [nm]} u_S(B)F(B)$

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for $S \subset [mn]$. Writing $S = (S_1, \ldots, S_n)$ with $S_1, \ldots, S_n \subset [m]$, and $B = (B_1, \ldots, B_n)$ similarly, we have $u_S(B) = \prod_{j=1}^n u_{S_j}(B_j)$. Hence

$$
\begin{aligned}
\hat{T}_i(S_1, \ldots, S_n) &= 2^{-mn} \sum_{B=(B_1, \ldots, B_n) \subset [mn]} T_i(B) u_{S_1}(B_1) \cdots u_{S_n}(B_n) \\
&= 2^{-mn} \sum_{B_i \subset [m]} T(B_i) u_{S_i}(B_i) \\
&= 2^{-mn+m} \hat{T}(S_i) = 2^{m(1-n)} |S_i|^{1/2}.
\end{aligned}
$$

Thus by (A.4) and (A.7), $\hat{W}_i(S_1, \ldots, S_n) = |S_i|^{1/2} \hat{\tilde{f}}(S_1, \ldots, S_n)$, so by Parseval's equation for functions on $\mathbb{Y}^n$,

$$
\begin{aligned}
\|W_i\|_2^2 &= \sum_{S_1, \ldots, S_n \subset [m]} (\hat{W}_i(S_1, \ldots, S_n))^2 \\
&= \sum_{S_1, \ldots, S_n \subset [m]} |S_i| \hat{\tilde{f}}(S_1, \ldots, S_n)^2. \tag{A.8}
\end{aligned}
$$

But also,

$$
\begin{aligned}
\|W_i\|_2^2 &= 2^{-mn} \sum_{S_1, \ldots, S_n} (v[S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n](S_i))^2 \\
&= 2^{-mn} 2^m \sum_{S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n} \|v[S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n]\|_2^2 \\
&= 2^{m(1-n)} \sum_{S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n} w(h[S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n])/4,
\end{aligned}
$$

where for the last line we have used (A.5). By (A.1),

$$
w(h[S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n]) \le 3k^2 q \log(4/q),
$$

and also $w(h[S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n]) = 0$ if $\tilde{f}(S_1, \ldots, S_{i-1}, \cdot, S_{i+1}, \ldots, S_n)$ is a constant function. Hence,

$$
\|W_i\|_2^2 \le (3/4) k^2 q \log(4/q) I_{f,\boldsymbol{p}}(i).
$$

Summing over i and combining with (A.8), we obtain that

$$
\sum_{\mathbf{S}=(S_1, \ldots, S_n)} \hat{\tilde{f}}(\mathbf{S})^2 \|\mathbf{S}\| \le (3/4) k^2 q \log(4/q) \sum_{i=1}^n \delta_i,
$$

where we set $\delta_i := I_{f,\boldsymbol{p}}(i)$ and $\|\mathbf{S}\| := \sum_{i=1}^n |S_i|$.

Let $\mathcal{S}_1 := \{\mathbf{S} : \|\mathbf{S}\| \ge 2k^2(t(1-t))^{-1} q \log(4/q) \sum_{i=1}^n \delta_i\}$. Then

$$
\begin{aligned}
\sum_{\mathbf{S} \in \mathcal{S}_1} \hat{\tilde{f}}(\mathbf{S})^2 &\le \frac{t(1-t) \sum_{\mathbf{S} \in \mathcal{S}_1} \|\mathbf{S}\| \hat{\tilde{f}}(\mathbf{S})^2}{2k^2 q \log(4/q) \sum_{i=1}^n \delta_i} \\
&\le \frac{3t(1-t)}{8}, \tag{A.9}
\end{aligned}
$$

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whereas by Parseval's equation, since $\hat{\tilde{f}}(\emptyset) = \mathbb{E}f(X) = h$ and $f(\cdot) \in \{0, 1\}$,

$$\sum_{\{\mathbf{S}:\|\mathbf{S}\|>0\}} \hat{\tilde{f}}(\mathbf{S})^2 = \|\tilde{f}\|_2^2 - (\mathbb{E}f(X))^2 = t(1-t). \tag{A.10}$$

Next, for $i \in [n]$ we define the function R_i on $\mathbb{Y}^n$ by

$$
\begin{aligned}
R_i &:= \sum_{S_1,\ldots,S_n \subset [m]:S_i \neq \emptyset} \hat{\tilde{f}}(S_1,\ldots,S_n)u_{S_1,\ldots,S_n} \\
&= \tilde{f} - \sum_{S_1,\ldots,S_{i-1},S_{i+1},\ldots,S_n \subset [m]} \hat{\tilde{f}}(S_1,\ldots,\emptyset,\ldots,S_n)u_{S_1,\ldots,\emptyset,\ldots,S_n}
\end{aligned}
$$

where we have used the Walsh-Fourier expansion of $\tilde{f}$, and where it is to be understood that the $\emptyset$ takes the place of S_i in the sequence $(S_1,\ldots,\emptyset,\ldots,S_n)$. Now,

$$\hat{\tilde{f}}(S_1,\ldots,\emptyset,\ldots,S_n) = \langle \tilde{f}, u_{(S_1\ldots,\emptyset,\ldots,S_n)} \rangle$$

$$
\begin{aligned}
&= 2^{-mn} \sum_{B_1,\ldots,B_n \subset [m]} \tilde{f}(B_1\ldots,B_n) \prod_{j:j \neq i} u_{S_j}(B_j) \\
&= 2^{m(1-n)} \sum_{B_1,\ldots,B_{i-1},B_{i+1},\ldots,B_n \subset [m]} g_i(B_1,\ldots,B_{i-1},B_{i+1},\ldots,B_n) \prod_{j:j \neq i} u_{S_j}(B_j),
\end{aligned}
$$

where we set $g_i(B_1,\ldots,B_{i-1},B_{i+1},\ldots,B_n)$ to be the value of $\tilde{f}(B_1,\ldots,B_n)$ averaged over all values of B_i. Hence

$$\hat{\tilde{f}}(S_1,\ldots,\emptyset,\ldots,S_n) = \hat{g}_i(S_1,\ldots,S_{i-1},S_{i+1},\ldots,S_n)$$

and so by a further Walsh-Fourier expansion, for any $B_1,\ldots,B_n \subset [m]$ we have

$$R_i(B_1,\ldots,B_n) = \tilde{f}(B_1,\ldots,B_n) - g_i(B_1,\ldots,B_{i-1},B_{i+1},\ldots,B_n).$$

Therefore $|R_i(\mathbf{B})| \leq 1$ for all $\mathbf{B} = (B_1,\ldots,B_n) \subset [mn]$, and $R_i(\mathbf{B}) = 0$ whenever $h[B_1,\ldots,B_{i-1},B_{i+1},\ldots,B_n]$, defined by (A.6), is a constant function. Therefore, writing $\|g\|_p$ for $(2^{-mn}\sum_{\mathbf{B} \subset [mn]}|g(\mathbf{B})|^p)^{1/p}$ for any real-valued function g defined on $\mathbb{Y}^n$ and any $p \geq 1$, we have that

$$\|R_i\|_{4/3}^{4/3} \leq I_{f,\boldsymbol{p}}(i),$$

and therefore by the Bonami-Beckner inequality (Lemma 4 of [5]), for $\varepsilon = 3^{-1/2}$,

$$\|T_\varepsilon R_i\|_2^2 \leq \|R_i\|_{1+\varepsilon^2}^2 \leq \delta_i^{3/2}, \tag{A.11}$$

where we set

$$T_\varepsilon R_i := \sum_{\mathbf{S} \subset [mn]} \hat{R}_i(\mathbf{S})\varepsilon^{|\mathbf{S}|}u_{\mathbf{S}}.$$

Since $\hat{R}_i(S_1,\ldots,S_n)$ is zero or $\hat{\tilde{f}}(S_1,\ldots,S_n)$, according to whether S_i is empty or not, so by Parseval's identity

$$\|T_\varepsilon R_i\|_2^2 = \sum_{\mathbf{S}=(S_1,\ldots,S_n)} \hat{\tilde{f}}(\mathbf{S})^2\varepsilon^{2\|\mathbf{S}\|}\mathbf{1}\{S_i \neq \emptyset\}. \tag{A.12}$$

For $\mathbf{S} = (S_1, \ldots, S_n) \subset [mn]$ let $\mu(\mathbf{S})$ denote the number of i such that $S_i \neq \emptyset$. Comparing (A.11) with (A.12) and summing over i yields

$$\sum_{\mathbf{S}} \hat{\hat{f}}(\mathbf{S})^2 \varepsilon^{2\|\mathbf{S}\|} \mu(\mathbf{S}) \leq \sum_{i=1}^{n} \delta_i^{3/2}.$$

Let $\mathcal{S}_2$ be the set of $\mathbf{S}$ such that $1 > \varepsilon^{2\|\mathbf{S}\|} \geq \left(2\sum_i \delta_i^{3/2}\right)/(t(1-t))$. Then

$$\sum_{\mathbf{S} \in \mathcal{S}_2} \hat{\hat{f}}(\mathbf{S})^2 \leq \sum_{\mathbf{S} \in \mathcal{S}_2} \frac{\mu(\mathbf{S})\varepsilon^{2\|\mathbf{S}\|} \hat{\hat{f}}(\mathbf{S})^2 t(1-t)}{2\sum_i \delta_i^{3/2}}$$
$$\leq t(1-t)/2.$$

Combined with (A.9) and (A.10), since $(3/8) + (1/2) < 1$, this shows that there exists $\mathbf{S}$ with $\|\mathbf{S}\| > 0$ lying neither in $\mathcal{S}_1$ nor in $\mathcal{S}_2$. Choosing such an $\mathbf{S}$, since $\mathbf{S} \notin \mathcal{S}_2$ we have $3^{-\|\mathbf{S}\|} < \left(2\sum_i \delta_i^{3/2}\right)/(t(1-t))$ so that

$$\|\mathbf{S}\| > \log\left(\frac{t(1-t)}{2\sum_i \delta_i^{3/2}}\right)/\log 3$$

but also $\mathbf{S} \notin \mathcal{S}_1$, so that

$$2k^2 q \log(4/q) \sum_i \delta_i > t(1-t) \log\left(\frac{t(1-t)}{2\sum \delta_i^{3/2}}\right)/\log 3.$$

Suppose (3.5) holds. Then, setting $\alpha := aq^2(\log(4/q))^2$ and $I_f := \sum_{i=1}^{n} \delta_i$, we have that $\delta_i \leq \alpha$ for all i so that $\sum_i \delta_i^{3/2} \leq \alpha^{1/2} I_f$. Since $2\log 3 < 3$, we have that

$$3k^2 I_f q \log(4/q) > t(1-t) \log\left(\frac{t(1-t)}{2\alpha^{1/2} I_f}\right).$$

Setting $x := t(1-t)/(q\log(4/q))$ and $b := I_f/x$, we have

$$I_f > \left(\frac{x}{3k^2}\right) \log\left(\frac{t(1-t)}{2a^{1/2}(q\log 4/q)bx}\right) = \left(\frac{x}{3k^2}\right) \log\left(\frac{1}{2a^{1/2}b}\right).$$

Since $I_f = bx$ it follows that $b \geq (1/3)k^{-2}\log(1/(2a^{1/2}b))$, and therefore $b+(1/3)k^{-2}\log b \geq (1/3)k^{-2}\log(1/(2a^{1/2}))$.

Since $(\log u)/u \leq e^{-1}$ for all $u > 0$, and since we assume $a \leq 1/16$ so that $\log(a^{-1/2}) \geq 2\log 2$, therefore

$$2b \geq b + (1/3)k^{-2}\log b \geq (1/3)k^{-2}\log(1/(2a^{1/2})) \geq (1/6)k^{-2}\log a^{-1/2}.$$

Therefore $b \geq (24k^2)^{-1}\log(1/a)$, which implies (3.6).